

ASYMPTOTICS OF ANALYTIC TORSION FOR HYPERBOLIC THREE-MANIFOLDS

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ABSTRACT. We prove that for certain sequences of nonuniform lattices in $\mathrm{SL}_2(\mathbb{C})$ which converge to \mathbb{H}^3 in a weak (“Benjamini-Schramm”) sense and certain coefficient systems the regularized analytic torsion approximates the L^2 -torsion of the universal cover under an additional hypothesis. In the arithmetic case we prove in addition an asymptotic equality between the former and some Reidemeister torsion of the truncated manifolds.

1. INTRODUCTION

1.1. Motivation: integral homology of congruence manifolds. In the paper [3] N. Bergeron and A. Venkatesh show that for odd m , in sequences of compact arithmetic hyperbolic m -manifolds which converge to \mathbb{H}^m the homological torsion has an exponential growth for certain local systems. That is, there exists \mathbb{Q} -representations of $\mathrm{SO}(m, 1)$ on a space V such that if Γ_n is a sequence of uniform arithmetic lattices in (the \mathbb{Q} -form used to define the representation of) $\mathrm{SO}(m, 1)$ such that the injectivity radius of the $M_n = \Gamma_n \backslash \mathbb{H}^m$ goes to infinity we have that

$$(1.1) \quad \liminf_{n \rightarrow \infty} \sum_{\substack{p=1, \dots, m-1 \\ p \equiv \frac{m-1}{2} \pmod{2}}} \frac{\log |H_p(\Gamma_n, V_{\mathbb{Z}})_{\mathrm{tors}}|}{\mathrm{vol} M_n} > 0.$$

In [1] it is proven that the above limit holds for any sequence of congruence subgroups of a uniform arithmetic lattice. Bergeron and Venkatesh conjecture that a more precise result should be true: let Γ be a congruence subgroup of $\mathrm{SO}(m, 1)$, then for any sequence Γ_n of distinct congruence lattices in a fixed \mathbb{Q} -form and any \mathbb{Q} -representation V one should actually have the limit

$$(1.2) \quad \lim_{n \rightarrow \infty} \frac{\log |H_{\frac{m-1}{2}}(\Gamma_n, V_{\mathbb{Z}})_{\mathrm{tors}}|}{\mathrm{vol} M_n} = C.$$

where C is an explicit positive (for odd m) constant depending on V . The main difficulty in proving this lies in the generalization to a general coefficient system, for the methods of [3] require an additional hypothesis on V (which is shown there to hold for many coefficient systems but fails in general, in particular for trivial ones). Also it is not clear at present how to prove that the torsion lies only in the middle degree homology. This is the first of two papers (the other one being the forthcoming [21]) originating from the author’s Ph.D. thesis [22]. where we study torsion for nonuniform congruence lattices in $\mathrm{SL}_2(\mathbb{C})$. Here we will deal with analytic torsion in the more general setting of finite-volume hyperbolic manifolds.

1.2. Analytic torsion and Cheeger-Müller equality. The main tools used in [3] are the Ray-Singer analytic torsion $T(M_n; V)$ and the Cheeger-Müller theorem. Bergeron and Venkatesh prove the limit

$$(1.3) \quad \lim_{n \rightarrow \infty} \frac{\log T(M_n; V)}{\mathrm{vol} M_n} = t^{(2)}(V)$$

and W. Müller's generalization [17] of the Cheeger-Müller Theorem then yields that

$$T(M_n; V) = \prod_{p=0}^m |H_p(\Gamma_n, V_{\mathbb{Z}})_{tors}|^{(-1)^p}$$

from which (1.1) follows at once since the L^2 -torsion $t^{(2)}(V)$ is positive for $n = 3 \pmod{4}$ and negative for $n = 1 \pmod{4}$. One of the issues in [1] is then to prove that (1.3) holds under weaker conditions than those of [3] and that these conditions are satisfied by sequences of congruence subgroups. Following the work of I. Benjamini and O. Schramm on graphs the notion of Benjamini-Schramm convergence of Riemannian manifolds is defined there (see (4.1) below) and it is then a relatively easy matter to show that the proofs of [3] extend to this setting.

It is not immediately clear how to adapt the scheme of proof of [3] to manifolds with cusps. The problem is that the Ray-Singer definition of analytic torsion does not extend to the general finite-volume setting. However one can define a “regularized” analytic torsion T_R (see 5 below, [19] or [16] for example) for which one can prove the following conditional approximation result.

Theorem A. *Let M be a finite-volume hyperbolic three-manifold and M_n a sequence of finite coverings of M which are BS-convergent to \mathbb{H}^3 and cusp-uniform. Let V be a finite-dimensional representation of $\mathrm{SL}_2(\mathbb{C})$ which is strongly acyclic, and suppose in addition that the intertwining operators are “well-behaved” near 0. Then we have the limit*

$$(1.4) \quad \lim_{n \rightarrow \infty} \frac{\log T_R(M_n; V)}{\mathrm{vol} M_n} = t^{(2)}(V).$$

In [21] we will check these conditions for congruence lattices. The constant $t^{(2)}(V)$ is equal to the quotient of the L^2 -torsion by the volume (see 5.13 below); its exact value has been computed in [3]:

$$(1.5) \quad t^{(2)}(V) = \frac{-1}{48\pi} ((n_1 + n_2 + 2)^3 - |n_1 - n_2|^3 + 3|n_1 - n_2|(n_1 + n_2 + 2)(n_1 + n_2 + 2 - |n_1 - n_2|)).$$

As for the analogue of the Cheeger-Müller theorem which would be the next step towards establishing exponential growth of torsion homology we deal with this only in the case where M is arithmetic, meaning that (the preimage in $\mathrm{SL}_2(\mathbb{C})$ of) $\pi_1(M)$ preserves a lattice in \mathbb{C}^2 . In this case we prove the asymptotic equality between regularized analytic torsion and the Reidemeister torsion for truncated manifolds in Theorem 6.2 below, which results in the following theorem (the truncated manifolds M^Y are defined in (2.3) and the absolute Reidemeister torsion τ_{abs} in (6.3) below).

Theorem B. *Suppose that M, M_n, V are as in the statement of the previous theorem and that $\pi_1(M)$ preserves a lattice $V_{\mathbb{Z}}$ in V . Then we have*

$$(1.6) \quad \lim_{n \rightarrow \infty} \frac{\log T_R(M_n; V) - \log \tau_{\mathrm{abs}}(M_n^{Y^n}, V_{\mathbb{Z}})}{\mathrm{vol} M_n} = 0.$$

In [21] this will be the first step towards establishing an asymptotic equality with a suitable Reidemeister torsion (defined using only the manifold M) for congruence subgroups and its application to homology growth.

1.3. Outline.

1.3.1. Convergence of finite-volume manifolds and regularized traces. In [1] the notion of BS-convergence and its implications for topological invariants are studied for cocompact lattices in general semisimple Lie groups. Here we extend some of these results to nonuniform lattices in $\mathrm{SL}_2(\mathbb{C})$. The notion of convergence is the same and it can actually be proven that—at least in

sequences of covers of a fixed manifold—the behaviour of unipotent elements does not matter for a sequence to converge, see 4.2.

On the other hand all our results are established under a condition on the geometry of the cusps of the sequence of manifolds, namely that they do not degenerate in the space of unimodular Euclidean tori (we call such sequences “cusp-uniform”), which allows us to estimate the series over the unipotent elements in a uniform manner.

The regularised trace $\text{Tr}_R(K)$ of an automorphic kernel K on a finite-volume manifold M is basically defined by taking either side of a very unrefined form of the trace formula for K , of which we give a mostly self-contained proof—minus the theory of Eisenstein series—in Section 3. The study of the geometric side in Benjamini-Schramm convergent sequences is not very hard and results in Theorem 4.4. We prove that the Betti numbers in a BS-convergent sequence are sublinear in the volume in Corollary 6.15—we cannot do this earlier because we did not manage to control the non-discrete part of the spectral side of the trace formula in general.

1.3.2. Analytic torsions. Our definition of analytic torsion for cusped manifolds is the same as in [19] or [16]¹ Let M be a finite-volume manifold and K_t^p its heat kernel on p -forms (with coefficients in a strongly acyclic bundle). One defines the analytic torsion as in the compact case, by putting:

$$T_R(M) = \sum_{p=0}^3 p(-1)^p \frac{d}{ds} \left(\frac{1}{\Gamma(s)} \int_0^{t_0} \text{Tr}_R(K_t^p) t^s \frac{dt}{t} \right)_{s=0} + \int_{t_0}^{+\infty} \text{Tr}_R(K_t^p) \frac{dt}{t}$$

(which does not depend on $t_0 > 0$); see 5 or [16] for the (many) details needed to ensure the convergence of the integral and their analytic continuation. In a sequence of finite covers we study the first summand using the geometric side of the trace formula and the second one using the spectral side, as in [3, Section 4]. The spectral side is dealt with using the uniform spectral gap property established there; however the part coming from the continuous spectrum causes some additional difficulty which explains the conditionality of our approximation on an hypothesis on intertwining operators which we were not able to check for general sequences. The study of the geometric side is actually quite simple once the asymptotic expansion for K_t^p at $t \rightarrow 0$ has been established (see Proposition 5.4) using our unrefined trace formula. We remark that in [22] we dealt with these problems in the more general context of finite-volume hyperbolic good orbifolds.

We also show that under the same hypotheses as for the approximation of analytic torsion there is an asymptotic equality between absolute torsion for the truncated manifold M^Y and regularized analytic torsion for the complete manifold, cf. Theorem 6.1 below. As in the proof of the approximation result we separate into small and large times. For small times we estimate on the integral of automorphic kernels over the truncated manifolds and a result of W. Lück and T. Schick [12]. For large times we need to control the spectral gap for the truncated manifolds and this is achieved using a result of F. Calegari and A. Venkatesh [7, Chapter 6].

1.3.3. Asymptotic Cheeger-Müller theorem and homology growth. In contrast with the compact case, for our coefficient systems there is usually a nontrivial homology in characteristic 0. Thus, to state and hopefully prove a Cheeger-Müller-type equality one needs to define a suitable Reidemeister torsion. This is done by F. Calegari and A. Venkatesh in [7], in a manner similar to the regularization for traces of integral operators. Thus a natural way to prove such an equality would be to apply the Cheeger-Müller equality for manifolds with boundary [6],[13] to the truncated manifolds and to compare both sides with their regularized analogue.

¹We could have just quoted the results of the latter but we use a slightly different method to prove the asymptotic expansion of the heat kernel which is better suited to the rest of this paper.

Here we deal only with the first part of this program, we refer to [21] for the applications of the results in the present paper to congruence subgroups and their homology growth. From the asymptotic equality of analytic torsions (Theorem 6.1) it is not hard to deduce an asymptotic equality with the absolute Reidemeister torsion of the truncated manifold using a recent generalization by J. Brüning and X. Ma of the Cheeger-Müller theorem, see Theorem 6.2.

1.4. Related recent results. In addition to the papers [3] and [1] from which this work originates there have been other papers dealing with similar problems. Following W. Müller's [18] there has been a number of papers studying the asymptotic behaviour of analytic torsion of a compact manifold as the coefficient systems varies, culminating with the work of J.M. Bismut, X. Ma and W. Zhang [4]. Extensions of this to the noncompact setting have been investigated in the work of P. Menal-Ferrer and J. Porti [15] and that of W. Müller and J. Pfaff [16]. The latter authors have told me that they are currently investigating the approximation problem for the analytic torsion using the methods of this last paper (namely studying a suitable trace formula, more refined than the one considered here). J. Pfaff has also proven an equality between *quotients* of Reidemeister torsions on one side and analytic ones on the other, see [20, Theorem 1.3]. Finally the book-to-appear [7] contains a study on the extension of the Jacquet-Langlands correspondance to torsion invariants of congruence hyperbolic 3-manifolds.

Acknowledgments. The reading of a preliminary version of [7] has been extremely profitable for the writing of this paper and I want to thank the authors for allowing me to read it. During the redaction I became more and more permeated by the point of view of Benjamini-Schramm convergence introduced in the joint work (with M. Abert, N. Bergeron, I. Biringer, T. Gelander, N. Nikolov and I. Samet) [1]. I also benefited greatly from a week spent in Bonn with W. Müller and J. Pfaff, whose comments on a previous version of this paper were especially useful. Last but not least I want to thank my Ph.D. advisor, Nicolas Bergeron, under whose supervision this work was conceived and written.

2. HYPERBOLIC MANIFOLDS AND THEIR LOCAL SYSTEMS

This section reviews more or less well-known stuff, with the additional purpose of fixing notations.

2.1. Height functions on \mathbb{H}^3 . Let $G = \mathrm{SL}_2(\mathbb{C})$ so that $K = \mathrm{SU}(2)$ is a maximal compact subgroup and the Riemannian symmetric space G/K is isometric to hyperbolic three-space, \mathbb{H}^3 , which we regard here as the Poincaré half-space $\mathbb{C} \times (0, +\infty)$ with the Riemannian metric given by $\frac{dzd\bar{z}+dy^2}{y^2}$ in coordinates (z, y)

Define the following subgroups of G :

$$P_\infty = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, a \in \mathbb{C}^\times, b \in \mathbb{C} \right\}, \quad N_\infty = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, b \in \mathbb{C} \right\},$$

$$A_\infty = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, a \in \mathbb{R}_+^\times \right\}, \quad M_\infty = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \theta \in [0, 2\pi] \right\}.$$

A parabolic subgroup of G is a subgroup conjugated to P_∞ . Let $P = gP_\infty g^{-1}$ be such a subgroup and N, A, M the conjugates of $N_\infty, A_\infty, M_\infty$ by g . We call norm on A any function that is conjugated by g to the function $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mapsto a^2$ on A_∞ . We have the Langlands decomposition $P = NAM = MAN$ and the Iwasawa decomposition $G = NAK$. A height function (on \mathbb{H}^3) at P is then defined to be any function of the form $gK \mapsto |a|$ where $g = nak \in NAK$ and $|\cdot|$ is any norm on A . (as an illuminating example take $P = P_\infty$, then the height functions at P are of the form $(z, y) \mapsto ty$ for $t \in \mathbb{R}_+^\times$).

The level sets of a height function at P are called horospheres through P ; they are isometric to the Euclidean plane \mathbb{C} and are acted upon simply transitively by the subgroup N . Let y_P be a height function at P , we identify N with $\{y_P = 1\} \cong \mathbb{C}$ and we denote by $|n|$ the induced length on N . We normalize the Haar measure dn on N so that it is the pullback of the Lebesgue measure on \mathbb{C} , then the volume form of \mathbb{H}^3 is equal to $dndy_P/y_P^3$. For $x \in \mathbb{H}^3$ the quotient $|n|/y_P(x)$ does not depend on the choice of y_P and we have the following estimate for the translation length of unipotent elements.

Lemma 2.1. *There exists a function $\ell : [0, +\infty) \rightarrow [0, +\infty)$ such that*

$$(2.1) \quad d(x, nx) = \ell \left(\frac{|n|}{y_P(x)} \right)$$

for all parabolics $P = MAN$, $n \in N$ and $x \in \mathbb{H}^3$. Moreover $\ell(r) \gg \log(1+r)$.

Proof. Obviously it suffices to prove the lemma for $P = P_\infty$; for $n = \begin{pmatrix} 1 & z \\ & 1 \end{pmatrix} \in N_\infty$ we may take $|n| = |z|$. Let $x \in \mathbb{H}^3$, $y = y_\infty(x)$. The formula [2, Corollaire A.5.8] yields

$$(2.2) \quad d(x, nx) = 2 \left(\log \left(1 + \sqrt{\frac{|n|^2}{|n|^2 + 4y^2}} \right) - \log \left(1 - \sqrt{\frac{|n|^2}{|n|^2 + 4y^2}} \right) \right)$$

so that $d(x, nx) = \ell(|n|/y)$ where we put

$$\ell(r) = 2 \left(\log \left(1 + (1 + (r/2)^{-2})^{-\frac{1}{2}} \right) - \log \left(1 - (1 + (r/2)^{-2})^{-\frac{1}{2}} \right) \right)$$

It remains to check that $\ell(r) \gg \log(1+r)$: the first summand is in $[0, \log(2)]$, and besides for $t \in [0, +\infty)$ one has $(1+t^2)^{-1/2} \geq (1+t)^{-1}$ so that

$$-\log(1 - (1+t^2)^{-1/2}) \geq -\log(1 - (1+t)^{-1}) = \log(1+t^{-1}).$$

from which the conclusion follows at once. \square

2.2. Height functions on hyperbolic three-manifolds. Let Γ be a lattice in G (i.e. Γ is discrete and $\Gamma \backslash G$ carries a finite right- G -invariant Borel measure). For any parabolic subgroup P we put $\Gamma_P = \Gamma \cap P$ and we say that P is Γ -rational if Γ_P contains a subgroup isomorphic to \mathbb{Z}^2 (equivalently $\Gamma \cap N$ is cocompact in N). Then Γ is cocompact if and only if there are no Γ -rational parabolics (equivalently if Γ contains no unipotent elements). In any case there are finitely many Γ -conjugacy classes of Γ -rational parabolics. We may thus choose representatives P_1, \dots, P_h for these classes and height functions y_{P_1}, \dots, y_{P_h} at each one of them, and put $y_j = \max_{\gamma \in \Gamma/\Gamma_{P_j}} y_{P_j}(\gamma^{-1}x)$ which we call a Γ -invariant height function (and which is, indeed, Γ -invariant). If Γ is torsion-free let M be the manifold $\Gamma \backslash \mathbb{H}^3$ and for $Y \in (0, +\infty)^h$ put :

$$(2.3) \quad M^Y = \{x \in M, \forall j = 1, \dots, h \text{ we have } y_j(x) \leq Y_j\}.$$

Then for Y large enough (depending on the choice of the original height functions y_{P_j}) M^Y is a compact manifold with boundary a union of flat tori T_j , $j = 1, \dots, h$. The ends $\{x \in M, y_j(x) \geq Y_j\}$ are isometric to the warped products $T_j \times (Y_j, +\infty)$ with the metrics $\frac{dx^2 + dy_j^2}{y_j^2}$ where dx^2 is the euclidean metric on T_j . In this paper we will always suppose that the height functions are normalized so that the maps $\Gamma_{P_j} \backslash \{y_{P_j} \geq 1\} \rightarrow M$ are embeddings.

Finally, if $\Gamma' \subset \Gamma$ is finite-index then note that the Γ -invariant height functions are Γ' -invariant and that all Γ' -invariant height functions are obtained in this way. When dealing with finite covers we will always suppose that the height functions come from those of the covered manifold.

2.3. Coefficient systems.

2.3.1. *Definitions.* Let $\Gamma \subset G$ be a lattice and put $M = \Gamma \backslash \mathbb{H}^3$. The flat (real) vector bundles (a.k.a. “real local systems”) on M are obtained as follows: if σ is a representation of Γ on a finite-dimensional real vector space W we get a vector bundle F_σ on M whose total space is the quotient $\Gamma \backslash (\mathbb{H}^3 \times W)$. For $\gamma \in \Gamma$ and a p -form f on \mathbb{H}^3 with coefficients in W ² we denote $\gamma^*f = \sigma(\gamma)^{-1} \circ f \circ \wedge^p T\gamma$. Then the p -forms on M with coefficients in F_σ correspond to Γ -equivariant sections of $\wedge^p T\mathbb{H}^3 \rightarrow V$ i.e. to those $f \in \Omega^p(\mathbb{H}^3; V)$ such that $\gamma^*f = f$ for all $\gamma \in \Gamma$.

Particularly interesting among all flat bundles are those whose holonomy comes from restricting a representation ρ of G on a real vector space V . The representation $\sigma = \rho|_\Gamma$ is never orthogonal but the bundle F_σ has an alternative description which yields a natural euclidean product. Up to scaling there is a unique inner product on V which is preserved by K and such that \mathfrak{P} acts by self-adjoint maps (ref.). We have a vector bundle E_ρ on M whose total space is $(\Gamma \backslash G \times V)/K$ so that it has a natural metric $|\cdot|$ coming from the K -invariant metric on V . The square-integrable sections of E_ρ correspond to the subspace:

$$\{f : \Gamma \backslash G \rightarrow V, |f| \in L^2(\Gamma \backslash G), \forall g \in G, k \in K, f(gk) = \rho(k^{-1})f(g)\}.$$

More generally, denoting by \mathfrak{p} the tangent space of \mathbb{H}^3 at the fixed point of K (which is an irreducible real K -representation) the square-integrable p -forms correspond to:

$$L^2\Omega^p(M; E_\rho) = (L^2(\Gamma \backslash G) \otimes V \otimes \wedge^p \mathfrak{p}^*)^K.$$

We have an isomorphism $E_\rho \rightarrow F_\sigma$ induced by the map $G \times V \rightarrow G \times V, (g, v) \mapsto (g, \rho(g).v)$. In the sequel we will denote by $L^2\Omega^p(M; V)$ the space of square-integrable p -forms on M with coefficients in E_ρ .

The Hodge Laplacians $\Delta^p[M]$ are essentially self-adjoint operators on the spaces $L^2\Omega^p(M; V)$, see [5] or [3, Section 3].

2.3.2. *Representations of $\mathrm{SL}_2(\mathbb{C})$.* Recall that the finite-dimensional representations of G are the natural representations ρ_{n_1, n_2} on the spaces

$$V_{n_1, n_2} = \mathrm{Sym}^{n_1}(\mathbb{C}^2) \otimes \mathrm{Sym}^{n_2} \overline{\mathbb{C}^2}$$

where G acts naturally on \mathbb{C}^2 and on $\overline{\mathbb{C}^2}$ by conjugate matrices. Note that $\rho_{n_1, n_2}(-1_G) = 1_{\mathrm{SL}(V_{n_1, n_2})}$ if and only if $n_1 - n_2$ is even. The most important (for us) feature of the representations V_{n_1, n_2} is the following spectral gap property, which is proven in [3, Lemma 4.1].

Proposition 2.2. *Let $n_1 \neq n_2$ and $V = V_{n_1, n_2}$. There exists $\lambda_0 > 0$ such that for any lattice Γ in G , $M = \Gamma \backslash \mathbb{H}^3$, $p = 0, 1, 2, 3$ and $\phi \in L^2\Omega^p(M; V)$ we have*

$$\langle \Delta^p[M]\phi, \phi \rangle_{L^2\Omega^p(M; V)} \geq \lambda_0 \|\phi\|_{L^2\Omega^p(M; V)}^2.$$

Bergeron and Venkatesh term this “strong acyclicity” of the representation.

2.4. **Euclidean lattices.** Let Λ be a lattice in \mathbb{C} ; we denote by $\mathrm{vol}(\Lambda)$ its covolume (i.e. the volume of a fundamental parallelogram) and put $\alpha(\Lambda) = \min\{|v|, v \in \Lambda, v \neq 0\}$. By Minkowski’s Theorem if $\pi r^2 \geq 4 \mathrm{vol}(\Lambda)$ then Λ contains a nonzero vector of length $\leq r$, which implies that the quotient $\mathrm{vol}(\Lambda)/\alpha(\Lambda)^2 \geq \pi/4$. We say that a set S of euclidean lattices is uniform if there exists a $C > 0$ such that

$$(2.4) \quad \forall \Lambda \in S, \mathrm{vol}(\Lambda) \leq C \alpha(\Lambda)^2$$

By Mahler’s criterion this is equivalent to asking that when we normalize the lattices in S so that they are unimodular they form a relatively compact set in $\mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$.

²i.e. f is a smooth section of the bundle $\wedge^p T^*\mathbb{H}^3 \otimes W$

We denote by $\mathcal{N}_\Lambda(r)$ the number of points in Λ of absolute value less than r and $\mathcal{N}_\Lambda^*(r) = \mathcal{N}_\Lambda(r) - 1^3$. The following estimate for the counting function was proven by Gauss; we include a proof only for the reader's convenience and because we need a precise statement with regard to the constants.

Lemma 2.3. *If S is a uniform set of lattices in \mathbb{C} then for any $\Lambda \in S$ we have*

$$(2.5) \quad E_\Lambda(r) := |\mathcal{N}_\Lambda^*(r) - \frac{\pi r^2}{\text{vol}(\Lambda)}| \ll \frac{r}{\alpha(\Lambda)}$$

with a constant depending only on S .

Proof. First we consider $r < \alpha(\Lambda)$ so that $E_\Lambda(r) = r^2/\text{vol} \Lambda$. By the uniformity of S we thus have $E_\Lambda(r) \ll (r/\alpha(\Lambda))^2 \leq r/\alpha(\Lambda)$ with a constant depending only on S .

Now suppose that $r \geq \alpha(\Lambda)$. By Minkowski's second theorem we can choose a fundamental parallelogram Π for Λ whose diameter d is $\asymp \frac{\text{vol}(\Lambda)}{\alpha(\Lambda)}$ (with absolute constants). For $r \geq d$ let z_1, \dots, z_n be the points in Λ with moduli $|z_k| \leq r$, then we have that $B(0, r-d) \subset \bigcup_k z_k + \Pi \subset B(0, r+d)$ so that $\pi(r-d)^2 \leq \text{vol}(\Pi)n \leq \pi(r+d)^2$. It follows that:

$$\left| \mathcal{N}_\Lambda(r) - \frac{\pi r^2}{\text{vol}(\Lambda)} \right| \leq \frac{d^2}{\text{vol}(\Lambda)} + \frac{2rd}{\text{vol}(\Lambda)} \ll \frac{\text{vol}(\Lambda)}{\alpha(\Lambda)^2} + \frac{r}{\alpha(\Lambda)}.$$

We have that $\frac{\text{vol}(\Lambda)}{\alpha(\Lambda)^2}$ is bounded by a constant depending only on S , so that the right-hand side is $\ll r/\alpha(\Lambda) + 1 \leq 2r/\alpha(\Lambda)$ which finishes the proof of (2.5) in this case. \square

3. REGULARIZED TRACES

Conventions. Throughout this whole section we fix a lattice $\Gamma \subset G$ such that for any Γ -rational parabolic $P = MAN$ we have $\Gamma_P \subset N$ (which forces $-1_G \notin \Gamma$). We also fix a representation ρ of G on a real vector space V . We choose representatives P_1, \dots, P_h of the conjugacy classes of Γ -rational parabolics and Γ -invariant height functions y_1, \dots, y_h such that the subsets $\{y_j \geq 1\} \subset \mathbb{H}^3$ are pairwise disjoint and their image in $M = \Gamma \backslash \mathbb{H}^3$ are embedded. We identify unipotent radicals of parabolics with horospheres at height one, which normalizes the Haar measures on the formers.

3.1. Spectral analysis on finite-volume manifolds. It is a well-known fact that one has the orthogonal sum

$$L^2\Omega^p(M; V) = L_{\text{disc}}^2\Omega^p(M; V) \oplus L_{\text{cont}}^2\Omega^p(M; V)$$

where $\Delta^p[M]$ has only discrete spectrum in $L_{\text{disc}}^2\Omega^p(M; V)$ and completely continuous spectrum in $L_{\text{cont}}^2\Omega^p(M; V)$. Here we briefly describe the proof of this result through the theory of Eisenstein series developed by Selberg, Langlands and others which actually yields a complete description of the continuous part.

3.1.1. Constant terms and cusp forms. Let P be any Γ -rational parabolic and $f \in L^2\Omega^p(\mathbb{H}^3; V)$ a Γ -equivariant p -form. Its constant term at P is defined to be the p -form given by

$$(3.1) \quad f_P(v) = \int_{\Gamma_P \backslash N} n^* f(v) \frac{dn}{\text{vol}(\Gamma_P \backslash N)}.$$

This descends a p -form on $\Gamma_P \backslash \mathbb{H}^3$ (which depends only on the Γ -conjugacy class of P) which is actually N -equivariant. If $h : \Gamma \backslash G \rightarrow V \otimes \wedge^p \mathfrak{p}^*$ is the K -equivariant function corresponding to f then the one corresponding to f_P is given by $g \mapsto 1/2 \int_{\Gamma_P \backslash N} h(ng) dn$. A p -form f is said to be cuspidal when $f_P = 0$ for all Γ -rational parabolics, and we denote by $L_{\text{cusp}}^2\Omega^p(M; V)$ the space of all such forms. Corollary 3.3 below implies that we have $L_{\text{cusp}}^2 \subset L_{\text{disc}}^2$.

³We will use \mathcal{N}^* rather than \mathcal{N} further on; moreover we get a cleaner bound in the lemma below.

3.1.2. *Eisenstein series.* If P is a Γ -rational parabolic there is a map E_P^p from the subspace of N -equivariant forms in $L^2\Omega^p(\mathbb{H}^3; V)$ to $L^2\Omega^p(M; V)$ given by

$$(3.2) \quad E_P^p(f) = \sum_{\gamma \in \Gamma/\Gamma_P} \gamma^* f.$$

If P, P' are two equivalent Γ -rational parabolics then the obvious map $\theta : L^2(N \setminus \mathbb{H}^3; V) \rightarrow L^2(N' \setminus \mathbb{H}^3; V)$ intertwines E_P^p and $E_{P'}^p$, i.e. $E_P^p = E_{P'}^p \circ \theta$. We choose representatives P_1, \dots, P_h of the conjugacy classes of Γ -rational parabolics and put $E^p = \bigoplus_{j=1}^h E_{P_j}^p$. Then we have the following facts:

- $\text{im}(E^p) = L_{\text{cusp}}^2\Omega^p(M; V)^\perp$;
- there is a finite-dimensional subspace L_{res}^2 inside $\text{im}(E)$ such that we have the orthogonal sum $\text{im}(E) = L_{\text{cont}}^2 \oplus L_{\text{res}}^2$.

When V is strongly acyclic the subspace L_{res}^2 is actually zero for all p ; when V is trivial it is of dimension one for $p = 0$ or $p = 3$ and zero for $p = 1, 2$.

We will now describe how the map allows to describe the continuous part $L_{\text{cont}}^2\Omega^p(M; V)$. We put $T_j = \Gamma_{P_j} \backslash N_j$ and identify $\Gamma_{P_j} \backslash \mathbb{H}^3$ with $T_j \times A_j$.

3.1.3. *Sections.* Let $v \in V^h$, one associates to it a section ϕ of the bundle induced by V on $\bigsqcup_j T_j$, defined for $n \in N_j$ by $\phi(n) = \rho(n).v_j$. For $s \in \mathbb{C}$ we may extend ϕ to a section ϕ_s of the bundle induced by $V_{\mathbb{C}}$ on $\bigsqcup_j \Gamma_{P_j} \backslash \mathbb{H}^3$ by putting $\phi_s(x) = y_j(x)^{s+1}\phi(n)$ for $x = na \in A_j N_j$. One then defines $E(s, v) = E(\phi_s)$ for $\text{Re}(s)$ large enough, which defines a holomorphic map from a half-plane to $C^\infty(M; V_{\mathbb{C}})$. A fundamental fact is that it may be analytically continued to a meromorphic map $\mathbb{C} \rightarrow C^\infty(M; V_{\mathbb{C}})$ which has at most a single simple pole at $s = 1$ and no other pole in the half-plane $\text{Re}(s) \geq 0^4$. There exists a meromorphic map

$$\Psi = (\Psi_1, \dots, \Psi_h) : \mathbb{C} \rightarrow \text{End}_{\mathbb{C}}(V_{\mathbb{C}})$$

having the same poles as $E(\cdot, v)$ and such that for all $s \in \mathbb{C}, v = (v_1, \dots, v_h) \in V^h$ one has

$$E(s, v)_{P_j} = y_j^{s+1} v_j + y_j^{1-s} \Psi_j(s)(v)$$

Moreover, $\Psi(s)^{-1} = \Psi(-s)$ and $\Psi(s)$ is unitary for $\text{Re}(s) = 0$. The section $E(s, v)$ is easily seen to be non-square integrable but for $\psi \in L^2(\mathbb{R})$ the section given by

$$E(\psi, v) := \int_{-\infty}^{+\infty} \psi(u) E(iu, v) du$$

is. The continuous part $L_{\text{cont}}^2(M; V_{\mathbb{C}})$ is spanned by these functions and one has the equality

$$(3.3) \quad \|E(\psi, v)\|_{L^2(M; V)} = \frac{1}{2\pi} \|\psi\|_{L^2(\mathbb{R})} |v|_{V^h}.$$

The space V decomposes as the orthogonal sum

$$V = \begin{cases} \bigoplus_{k=-q}^q V_{2k} & q = \frac{n_1+n_2}{2}, n_1 - n_2 \text{ pair}; \\ \bigoplus_{k=-q}^q V_{2k+1} & q = \frac{n_1+n_2-1}{2}, n_1 - n_2 \text{ impair} \end{cases}$$

where V_l is the subspace on which the abelian subgroup $M \cong \mathbb{S}^1$ acts by the character $z \mapsto z^l$. For $v \in (V_l)^h$ the sections $E(s, v)$ are eigenfunctions of the laplacian $\Delta^p[M]$ with eigenvalue $(1 - \frac{|l|}{2})^2 - s^2 + \lambda_V$ (where λ_V is the Casimir eigenvalue of V , $\lambda_{V_{n_1, n_2}} > 0$ if $n_1 \neq n_2$).

⁴When V is strongly acyclic it is holomorphic in this closed half-plane

3.1.4. *1-forms.* The space $\Omega^+(V)$ of holomorphic 1-forms on $\bigsqcup_j T_j$ with coefficients in V is isomorphic to V^h , as is the space $\Omega^-(V)$ of anti-holomorphic 1-forms. Any $\omega \in \Omega^\pm(V)$ may be extended to a 1-form ω_s on $\bigsqcup_j \Gamma_{P_j} \backslash \mathbb{H}^3$ by putting $\omega_s(x) = y_j(x)^{s+1} a^* \omega(n)$ for $x = na \in N_j A_j$. The series $E(s, \omega) = E(\omega_s)$ is convergent $\operatorname{Re}(s)$ large enough and is analytically continued to an entire function from \mathbb{C} to $\Omega^1(M; V_{\mathbb{C}})$. There are holomorphic maps

$$\Phi^\pm = (\Phi_1^\pm, \dots, \Phi_h^\pm) : \mathbb{C} \rightarrow \operatorname{Hom}_{\mathbb{C}}(\Omega^\pm(V), \Omega^\mp(V))$$

such that for all $s \in \mathbb{C}, \omega = (\omega_1, \dots, \omega_h) \in \Omega^\pm(V)$ we have

$$E(s, \omega)_{P_j} = y_j^s \omega_j + y_j^{-s} \Phi_j^\pm(s)(\omega)$$

Also, $\Phi^\pm(s)^{-1} = \Phi^\mp(-s)$ and $\Phi^\pm(s)$ is unitary for $\operatorname{Re}(s) = 0$. As above $E(s, \omega)$ is not square integrable but if we put

$$E(\varphi, \omega) = \int_{-\infty}^{+\infty} \varphi(u) E(iu, \omega) du$$

for $\varphi \in L^2(\mathbb{R})$ it is, and the subspace $L_{\text{cont}}^2 \Omega^1(M; V_{\mathbb{C}})$ is spanned by those. Moreover,

$$(3.4) \quad \|E(\varphi, \omega)\|_{L^2 \Omega^1(M; V)} = \frac{1}{2\pi} \|\varphi\|_{L^2(\mathbb{R})} |\omega|_{\Omega^\pm(V)}.$$

Again, the spaces Ω^\pm decompose as orthogonal sums

$$\begin{aligned} \Omega^+(V) &= \begin{cases} \bigoplus_{k=-q}^q \Omega^+(V_{2k+2}) & q = \frac{n_1+n_2}{2}, n_1 - n_2 \text{ even}; \\ \bigoplus_{k=-q}^q \Omega^+(V_{2k+3}) & q = \frac{n_1+n_2+1}{2}, n_1 - n_2 \text{ odd}; \end{cases} \\ \Omega^-(V) &= \begin{cases} \bigoplus_{k=-q}^q \Omega^-(V_{2k-2}) & q = \frac{n_1+n_2}{2}, n_1 - n_2 \text{ even}; \\ \bigoplus_{k=-q}^q \Omega^-(V_{2k-1}) & q = \frac{n_1+n_2+1}{2}, n_1 - n_2 \text{ odd}; \end{cases} \end{aligned}$$

and for $\omega \in \Omega^\pm(V_l)$ the 1-form $E(s, \omega)$ is an eigenform of the laplacian with eigenvalue $(1 - \frac{|l|}{2})^2 - s^2 + \lambda_V$.

3.1.5. *2- and 3-forms.* The Hodge $*$ yields isometries $L^2 \Omega^p(M; V) \rightarrow L^2 \Omega^{3-p}(M; V^*)$, so that the spectral decomposition for $L^2 \Omega^2, L^2 \Omega^3$ spaces follows from that of $L^2 \Omega^1$ and L^2 respectively.

3.1.6. *Truncated forms and Maass-Selberg relations.* One can “truncate” forms on M by subtracting their constant term from a certain height Y on. Formally, for $Y = (Y_1, \dots, Y_h) \in [1, +\infty)$ one defines the truncation operator at height Y by:

$$T^Y f(g) = f(g) - \sum_{j=1}^h 1_{[Y_j, +\infty)}(y_j(g)) f_{P_j}(g)$$

for $f \in \Omega^p(M; V)$. The truncated Eisenstein series are square integrable on M and one can actually compute their norm:

$$(3.5) \quad \begin{aligned} \|T^Y E(s, v)\|_{L^2(M; V)}^2 &= 2 \sum_{j=1}^h \log(Y_j) |v_j|_V^2 + \langle \Psi(s)^{-1} \Psi(s)'(v), v \rangle_{V^h} \\ &\quad + \sum_{j=1}^h \frac{1}{s} (Y_j^s \langle \Psi_j(-s)(v), v_j \rangle_V - Y_j^{-s} \langle \Psi_j(s)(v), v_j \rangle_V) \end{aligned}$$

$$(3.6) \quad \|T^Y E(s, \omega)\|_{L^2 \Omega^1(M; V)}^2 = 2 \sum_{j=1}^h \log(Y_j) |\omega_j|_{\Omega^\pm(V)}^2 + \langle \Phi^\pm(s)^{-1} \Phi^\pm(s)'(\omega), \omega \rangle_{\Omega^\pm(V)}.$$

3.2. Regularized trace of automorphic kernels.

3.2.1. *Automorphic kernels.* The laplacians on \mathbb{H}^3 are essentially self-adjoint operators so that for $\phi \in C^\infty([0, +\infty))$ one may define $\phi(\Delta^p[\mathbb{H}^3])$ on $L^2\Omega^p(\mathbb{H}^3; V)$. For $\phi \in \mathcal{S}(\mathbb{R})$ this operator is given by convolution with a kernel

$$k_{\phi,p} \in C^\infty(\mathbb{H}^3 \times \mathbb{H}^3; (\wedge^p T\mathbb{H}^3 \otimes V) \otimes (\wedge^p T\mathbb{H}^3 \otimes V)^*),$$

i.e. $k_{\phi,p}(x, y) \in \text{Hom}(\wedge^p T_x^* \mathbb{H}^3 \otimes V, \wedge^p T_y^* \mathbb{H}^3 \otimes V)$ and for a p -form $f \in L^2\Omega^p(\mathbb{H}^3; V)$ one has

$$\phi(\Delta^p[\mathbb{H}^3])f(y) = \int_{\mathbb{H}^3} k_{\phi,p}(x, y)f(x)dx.$$

The kernels $k_{\phi,p}$ are invariant under the diagonal action of G on $\mathbb{H}^3 \times \mathbb{H}^3$, meaning that for $g \in G$, $x, y \in \mathbb{H}^3$ we have

$$(3.7) \quad k_{\phi,p}(x, y) = (\wedge^p T_x g^{-1} \otimes \text{Id}_V) \circ k_{\phi,p}(gx, gy) \circ (\wedge^p T_x g \otimes \text{Id}_V).$$

The Plancherel formula for G allows to compute k_ϕ which results in the following lemma.

Lemma 3.1. *There exists a C^∞ -dense subset $\mathcal{A}(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})$ such that for any $\phi \in \mathcal{A}(\mathbb{R})$ we have $k_{\phi,p}(x, y) \ll e^{-Ad(x,y)}$ for all $A > 0$.*

For $g \in G$ put $g^*k_{\phi,p}(x, y) = (\wedge^p T_x g^{-1} \otimes \rho(g)^{-1}) \circ k_{\phi,p}(x, y) \in \text{Hom}(\wedge^p T_x \mathbb{H}^3 \otimes V, \wedge^p T_{g^{-1}y} \mathbb{H}^3 \otimes V)$. By the above Lemma we have $|g^*k_{\phi,p}(x, y)| \ll e^{-Ad(x,y)}$ so that by the well-known estimate

$$(3.8) \quad |\{\gamma \in \Gamma, d(x, \gamma y) \leq r\}| \leq Ce^{cr}$$

(where c is absolute and C depends on Γ) the following series converges uniformly on compact sets of $\mathbb{H}^3 \times \mathbb{H}^3$

$$K_{\phi,p}^\Gamma(x, y) = \sum_{\gamma \in \Gamma} \gamma^* k_{\phi,p}(x, \gamma y).$$

The kernel $K_{\phi,p}^\Gamma$ is Γ -equivariant in each variable and for $f \in L^2\Omega^p(M; V)$ we have

$$(3.9) \quad \phi(\Delta^p[M])f(y) = \int_{\Gamma \backslash \mathbb{H}^3} K_{\phi,p}^\Gamma(x, y)f(x)dx.$$

3.2.2. *Truncation.* in the sequel we abusively write $k_{\phi,p}f = \phi(\Delta^p)f$, etc. Let P be a parabolic subgroup, we define the constant term of $k_{\phi,p}$ to be the kernel given by

$$(k_{\phi,p})_P(x, y) = \int_N n^* k_{\phi,p}(x, ny)dn.$$

For a Γ -rational P we define the constant term $(K_{\phi,p}^\Gamma)_P$ of $K_{\phi,p}^\Gamma$ at P by

$$(K_{\phi,p}^\Gamma)_P(x, y) = \sum_{\gamma \in \Gamma_P \backslash \Gamma} \frac{1}{\text{vol}(\Gamma_P \backslash N)} \gamma^* (k_{\phi,p})_P(x, \gamma y)$$

For $f \in L^2\Omega^p(M; V)$ a routine calculation yields

$$(3.10) \quad (K_{\phi,p}^\Gamma)_P(f) = (K_{\phi,p}^\Gamma)(f_P)$$

One naturally defines the truncated kernel on M by

$$T^Y K_{\phi,p}^\Gamma = \begin{cases} K_{\phi,p}^\Gamma(x, y) - \sum_{j=1}^h (K_{\phi,p}^\Gamma)_{P_j}(x, y) & y_j(y) \geq Y_j; \\ K_{\phi,p}^\Gamma(x, y) & y \in M^Y \end{cases}$$

and it follows from (3.10) that

$$(3.11) \quad T^Y K_{\phi,p}^\Gamma(f) = K_{\phi,p}^\Gamma(T^Y f).$$

We give a somewhat detailed proof of the following well-known proposition (for a proof in the case of functions see [9, Chapitre 6.5]) because we will use it to get the geometric expression for the regularized trace.

Proposition 3.2. *For $\phi \in \mathcal{A}(\mathbb{R})$ the operator given by convolution with the kernel $T^Y K_{\phi,p}^\Gamma$ is trace-class.*

Proof. By classical arguments (see [22, Proposition 3.1]) we only need to show that the function

$$x \mapsto \text{tr} \left(K_{\phi,p}^\Gamma(x, x) - \sum_{j=1}^h (K_{\phi,p})_{P_j}^\Gamma(x, x) \right)$$

is integrable on M , i.e. that its integral on the ends $M - M^Y$ is absolutely convergent. We fix a fundamental domain D for the action of Γ on \mathbb{H}^3 whose vertices at infinity are exactly the fixed points of P_1, \dots, P_h and for $Y \in [1, +\infty)^h$ we put

$$D^Y = \{x \in D, \forall j y_j(x) \leq Y_j\} = \{x \in D, \forall j y_{P_j}(x) \leq Y_j\}$$

(note that y_j and y_{P_j} agree high in the cusp but of course not on \mathbb{H}^3). We have now

$$\begin{aligned} \int_{M-M^Y} \text{tr} K_{\phi,p}^\Gamma(x, x) dx &= \int_{D-D^Y} \text{tr}(K_{\phi,p}^\Gamma - (K_{\phi,p})_P)(x, x) dx \\ &= \sum_{j=1}^h \text{vol } \Lambda_j \int_Y^{+\infty} \left(\sum_{\eta \in \Lambda_j - \{0\}} h(\ell(|\eta|/y)) - \frac{1}{\text{vol } \Lambda_j} \int_{N_j} h(\ell(|n|/y)) dn \right) \frac{dy}{y^3} \\ &\quad + \int_{D-D^Y} \left(\sum_{\gamma \in \Gamma_{\text{lox}}} \text{tr}(\gamma^* k_{\phi,p}(x, \gamma x)) + \sum_{j=1}^h \sum_{\gamma \in \Gamma/\Gamma_{P_j}} \sum_{\eta \in \Lambda_j - \{0\}} \text{tr}(\gamma^* k_{\phi,p}(x, \gamma \eta \gamma^{-1} x)) \right) dx \\ &\quad - \sum_{j=1}^h \int_{D-D^Y} \frac{1}{\text{vol } \Lambda_j} \sum_{\gamma \in \Gamma/\Gamma_{P_j}} \int_{N_j} h(\ell(|n|/y_{P_j}(\gamma^{-1} x))) dn dx. \end{aligned}$$

It is well-known (see *loc. cit.*) that the integrals over D of the integrands on the second and third lines are absolutely convergent so that they go to 0 as $Y \rightarrow \infty$. It remains to see that

$$\int_Y^{+\infty} \left(\sum_{\eta \in \Lambda_j - \{0\}} h(\ell(|\eta|/y)) - \frac{1}{\text{vol } \Lambda_j} \int_{N_j} h(\ell(|n|/y)) dn \right) \frac{dy}{y^3}$$

is absolutely convergent. To see that we integrate by parts:

$$\begin{aligned} &\int_Y^{+\infty} \left(\sum_{\eta \in \Lambda_j - \{0\}} h(\ell(|\eta|/y)) - \frac{1}{\text{vol } \Lambda_j} \int_{N_j} h(\ell(|n|/y)) dn \right) \frac{dy}{y^3} \\ &= \int_Y^{+\infty} \int_0^{+\infty} h(\ell(r/y)) (d\mathcal{N}_{\Lambda_j}^*(r) - \frac{2\pi r}{\text{vol } \Lambda_j} dr) \frac{dy}{y^3} \\ &= \int_Y^{+\infty} \int_0^{+\infty} \frac{dh(\ell(r/y))}{dr} \left(\mathcal{N}_{\Lambda_j}^*(r) - \frac{\pi r^2}{\text{vol } \Lambda_j} \right) dr \frac{dy}{y^3} \\ &= \int_Y^{+\infty} \int_0^{+\infty} \frac{dh(\ell(r))}{dr} E_{\Lambda_j}(ry) dr \frac{dy}{y^3}. \end{aligned}$$

We have $E_{\Lambda_j}(ry) \ll y$ as $y \rightarrow \infty$ so that the integral is indeed convergent. \square

The following result implies the fact that $L_{\text{cusp}}^2 \subset L_{\text{disc}}^2$ (actually it would have sufficed to show that $\phi(\Delta_{\text{cusp}}^p[M])$ is Hilbert-Schmidt).

Corollary 3.3. *For $\phi \in \mathcal{A}(\mathbb{R})$ the operators $\phi(\Delta_{\text{cusp}}^p[M])$ and $\phi(\Delta_{\text{disc}}^p[M])$ are trace-class.*

Proof. As L_{cusp}^2 has finite codimension in L_{disc}^2 it suffices to prove that $\phi(\Delta_{\text{cusp}}^p[M])$ is trace-class. Since $y \mapsto K_{\phi,p}^\Gamma(x, y)$ is orthogonal to $L_{\text{cusp}}^2 \Omega^p(M; V)$ we get that $T^Y K_{\phi,p}^\Gamma f = K_{\phi,p}^\Gamma f$ for all $f \in L_{\text{cusp}}^2 \Omega^p(M; V)$ and the fact that $\phi(\Delta_{\text{cusp}}^p[M])$ be trace-class follows from the fact that $T^Y K_{\phi,p}^\Gamma$ itself is trace-class as well as the truncated operator $\phi(\Delta_{\text{Eis}}^p[M])$. \square

3.2.3. Spectral expansion. For $\phi, \psi \in L^2(\mathbb{R})$, $v \in V_l$ and $\omega \in \Omega^\pm(V_l)$ we get the identities

$$\begin{aligned} K_{\phi,p}^\Gamma E(\psi, \omega) &= \int_{-\infty}^{+\infty} \phi((1 - \frac{|l|}{2})^2 + u^2 + \lambda_V) \psi(u) E(iu, \omega) du \\ K_{\phi,p}^\Gamma E(\varphi, v) &= \int_{-\infty}^{+\infty} \phi((1 - \frac{|l|}{2})^2 + u^2 + \lambda_V) \varphi(u) E(iu, v) du. \end{aligned}$$

Put $d = \dim V$, choose orthonormal bases v_k , $k = 1, \dots, dh$ for V and $\omega_j, \bar{\omega}_j$, $j = 1, \dots, dh$ for $\Omega^\pm(V)$ (where all $v_k \in (V_k)^h$ and $\omega_j, \bar{\omega}_j \in \Omega^\pm(V_j)$). From the preceding identities, (3.3), (3.4) and (3.11) it follows that

$$\begin{aligned} \text{Tr}(T^Y (K_{\phi,0}^\Gamma)_{\text{Eis}}) &= \int_{-\infty}^{+\infty} \sum_{k=1}^{dh} \phi((1 - \frac{|l_k|}{2})^2 + u^2 + \lambda_V) \|T^Y E(iu, v_k)\|^2 du \\ \text{Tr}(T^Y (K_{\phi,1}^\Gamma)_{\text{Eis}}) &= \text{Tr}(T^Y (K_{\phi,0}^\Gamma)_{\text{Eis}}) + \frac{1}{2\pi} \int_{-\infty}^{+\infty} \sum_{j=1}^{dh} \phi((1 - \frac{|l_j|}{2})^2 + u^2 + \lambda_V) 2 \|T^Y E(iu, \omega_j)\|^2 du \end{aligned}$$

Putting $\Phi(s) = \Phi^-(s)\Phi^+(s)$ one sees that

$$\text{tr}(\Phi(s)^{-1}\Phi(s)') = \text{tr}(\Phi^+(s)^{-1}\Phi^+(s)') + \text{tr}(\Phi^-(s)^{-1}\Phi^-(s)')$$

and the Maass-Selberg relations (3.5), (3.6) yield (for even $n_1 - n_2$, $q = \frac{n_1+n_2}{2}$ and $d_l = \dim(V_l)$, Ψ_l is the restriction of Ψ to V_l , etc.)

(3.12)

$$\begin{aligned} \text{Tr}(T^Y K_{\phi,0}^\Gamma) &= \text{Tr}(K_{\phi,0}^\Gamma)_{\text{disc}} - \frac{1}{2\pi} \int_{-\infty}^{+\infty} \sum_{l=-2q}^{2q} \phi((1 - \frac{|l|}{2})^2 + u^2 + \lambda_V) \text{tr}(\Psi_l(iu)^{-1} \frac{d\Psi_l(iu)}{du}) du \\ &\quad + \sum_{j=1}^h \frac{\log Y_j}{\pi} \int_{-\infty}^{+\infty} \sum_{l=-2q}^{2q} d_l \phi((1 - \frac{|l|}{2})^2 + u^2 + \lambda_V) du \\ &\quad + \frac{1}{2\pi} \sum_{j=1}^h \sum_{l=-2q}^{2q} \phi((1 - \frac{|l|}{2})^2 + u^2 + \lambda_V) \int_{-\infty}^{+\infty} \frac{Y_j^{iu} \text{tr} \Psi(-iu) - Y_j^{-iu} \text{tr} \Psi(iu)}{iu} du \end{aligned}$$

(3.13)

$$\begin{aligned} \text{Tr}(T^Y K_{\phi,1}^\Gamma) &= \text{Tr}(T^Y (K_{\phi,0}^\Gamma)_{\text{Eis}}) + \sum_{j=1}^h \frac{2 \log Y_j}{\pi} \int_{-\infty}^{+\infty} \sum_{l=-2q-2}^{2q+2} d_l \phi((1 - \frac{|l|}{2})^2 + u^2 + \lambda_V) du \\ &\quad + \text{Tr}(K_{\phi,1}^\Gamma)_{\text{disc}} - \frac{1}{2\pi} \int_{-\infty}^{+\infty} \sum_{l=-2q-2}^{2q+2} \phi((1 - \frac{|l|}{2})^2 + u^2 + \lambda_V) \text{tr}(\Phi_l(iu)^{-1} \frac{d\Phi_l(iu)}{du}) du. \end{aligned}$$

3.2.4. *Regularized trace at last.* The expansions (3.12) and (3.13) allow to define regularized traces by subtracting the diverging part when $Y \rightarrow \infty$, for example in degree 0 we put :

$$(3.14) \quad \mathrm{Tr}_R K_{\phi,0}^\Gamma = \lim_{Y \rightarrow +\infty} \left(\mathrm{Tr}(T^Y K_\phi^\Gamma) - \sum_{j=1}^h \frac{h \log Y_j}{\pi} \int_{-\infty}^{+\infty} \sum_{l=-2q}^{2q} d_l \phi \left(\left(1 - \frac{|l|}{2}\right)^2 + u^2 + \lambda_V \right) du \right)$$

A classical computation (cf. [9, Proposition 5.3 in Chapter 6]) shows that for any function $\xi \in \mathcal{S}(\mathbb{R})$ one has

$$\lim_{Y \rightarrow \infty} \left(\int_{-\infty}^{+\infty} \xi(u) \frac{Y^{2iu} \mathrm{tr} \Psi(-iu) - Y^{-2iu} \mathrm{tr} \Psi(iu)}{2iu} du \right) = \frac{1}{4} \xi(0) \mathrm{tr} \Psi(0)$$

and (3.13) thus yields the spectral equality

$$(3.15) \quad \begin{aligned} \mathrm{Tr}_R K_{\phi,0}^\Gamma &= \mathrm{Tr}(K_{\phi,0}^\Gamma)_{\mathrm{disc}} + \frac{1}{4} \sum_{l=-2q}^{2q} d_l \phi \left(\left(1 - \frac{|l|}{2}\right)^2 + \lambda_V \right) \mathrm{tr} \Psi_l(0) \\ &\quad - \frac{1}{2\pi} \int_{-\infty}^{+\infty} \sum_{l=-2q}^{2q} d_l \phi \left(\left(1 - \frac{|l|}{2}\right)^2 + u^2 + \lambda_V \right) \mathrm{tr} \left(\Psi_l(iu)^{-1} \frac{d\Psi_l(iu)}{du} \right) du. \end{aligned}$$

We leave it to the reader to see what the definitions and spectral expansions for other degrees and for odd $n_1 - n_2$ exactly are. These expressions show that the regularized trace does not depend on the choice of height functions (which does not follow immediately from the definition).

3.3. **Geometric side.** For N a unipotent subgroup, $n \in N$ and $x \in \mathbb{H}^3$ the trace $\mathrm{tr}(n^* k_\phi(x, nx))$ depends only on $\ell = d(x, nx)$ and we thus put

$$(3.16) \quad h(\ell) = \mathrm{tr}(n^* k_\phi(x, nx)).$$

Proposition 3.4. *Let Λ_j be the Euclidean lattice $\Gamma_{P_j} \subset N_j$. We have*

$$\begin{aligned} \mathrm{Tr}_R K_\phi^\Gamma &= \mathrm{Tr}_\Gamma k_\phi + \int_D \sum_{\gamma \in \Gamma_{\mathrm{lox}}} \mathrm{tr}(\gamma^* k_\phi(x, \gamma x)) dx \\ &\quad + 2\pi h \int_0^{+\infty} r \log(r) h(\ell(r)) dr + \sum_{j=1}^h \kappa_j \mathrm{vol} \Lambda_j \int_0^{+\infty} r h(\ell(r)) dr \end{aligned}$$

where

$$\kappa_j = 2 \int_{\alpha(\Lambda_j)}^{+\infty} E_{\Lambda_j}(\rho) \frac{d\rho}{\rho^3} - \frac{\pi(1 + 2 \log \alpha(\Lambda_j))}{\mathrm{vol} \Lambda_j}.$$

Proof. We will show that there is an asymptotic expansion of the form

$$\int_{M^Y} \mathrm{tr} K_{\phi,p}^\Gamma(x, x) dx = A \log Y + B + o(1).$$

Since we have $\mathrm{Tr} T^Y K_{\phi,p}^\Gamma - \int_{M^Y} \mathrm{tr} K_{\phi,p}^\Gamma(x, x) dx = o(1)$ (see the proof of Proposition 3.2) it follows from this that we have $\mathrm{Tr}_R K_\phi^\Gamma = B$ and the computation of B will yield the result. Let D, D^Y be as in the proof of Proposition 3.2. Let $P = MAN$ be a parabolic and Λ be a Euclidean lattice in N . Let y_P be a height function at P and O is a fundamental domain for Λ acting on the horosphere $\{y_P = 1\}$ and put

$$(3.17) \quad B_\Lambda = AO, \quad B_\Lambda^Y = \{x \in B_\Lambda, y_P(x) \leq Y\}.$$

We want to get an asymptotic expansion when $Y \rightarrow \infty$ of:

$$\int_{B_\Lambda^Y} \sum_{\eta \in \Lambda - \{0\}} h(d(x, \eta x)) dx.$$

The integrand is N -invariant so that the integral equals:

$$\begin{aligned} \text{vol}(\Lambda) \int_0^Y \sum_{\eta \in \Lambda - \{0\}} h(\ell(|\eta|/y)) \frac{dy}{y^3} &= \text{vol}(\Lambda) \sum_{\eta \in \Lambda - \{0\}} \int_{|\eta|/Y}^{+\infty} h(\ell(r)) \frac{r dr}{|\eta|^2} \\ &= \text{vol}(\Lambda) \int_0^{+\infty} r h(\ell(r)) \sum_{0 < |\eta| \leq rY} \frac{1}{|\eta|^2} dr. \end{aligned}$$

Recall that \mathcal{N}_Λ is the counting function for the lattice Λ , $\mathcal{N}_\Lambda^* = \mathcal{N}_\Lambda - 1$ and $E_\Lambda(r) = \mathcal{N}_\Lambda^* - \frac{\pi r^2}{\text{vol} \Lambda} \ll \frac{r}{\alpha(\Lambda)}$, so that for $R > 0$ we have:

$$\begin{aligned} \sum_{0 < |v| \leq R} \frac{1}{|v|^2} &= \int_{\alpha(\Lambda)}^R \frac{d\mathcal{N}_\Lambda^*(\rho)}{\rho^2} = \frac{\mathcal{N}_\Lambda^*(R)}{R^2} + \int_{\alpha(\Lambda)}^R 2 \frac{\mathcal{N}_\Lambda^*(\rho)}{\rho^3} d\rho \\ &= \frac{\mathcal{N}_\Lambda^*(R)}{R^2} + \frac{2\pi(\log(R) - \log \alpha(\Lambda))}{\text{vol}(\Lambda)} + \int_{\alpha(\Lambda)}^{+\infty} E_\Lambda(\rho) \frac{d\rho}{\rho^3} - \int_R^{+\infty} E_\Lambda(\rho) \frac{d\rho}{\rho^3}. \end{aligned}$$

Putting

$$\kappa_\Lambda = \int_{\alpha(\Lambda)}^{+\infty} E_\Lambda(\rho) \frac{d\rho}{\rho^3} - \frac{\pi(1 + 2 \log \alpha(\Lambda))}{\text{vol} \Lambda}$$

we thus get

$$\begin{aligned} (3.18) \quad \int_{B_\Lambda^Y} \sum_{\eta \in \Lambda - 0} h(d(x, \eta x)) dx &= \int_0^{+\infty} 2\pi \log(rY) r h(\ell(r)) dr + \kappa_\Lambda \text{vol} \Lambda \int_0^{+\infty} h(\ell(r)) dr \\ &\quad - \int_0^{+\infty} r h(\ell(r)) \int_{\max(\alpha(\Lambda), rY)}^{+\infty} E_\Lambda(\rho) \frac{d\rho}{\rho^3} d\rho + \int_0^{+\infty} \frac{E_\Lambda(rY)}{r^2 Y^2} r h(\ell(r)) dr. \end{aligned}$$

The second line tends to 0 as $Y \rightarrow \infty$. Finally, applying (3.18) this to each $\Lambda_j := \Gamma_{P_j}$ and summing over the cusps we get the asymptotic expansion

$$\begin{aligned} \int_{M^Y} \text{tr} K_{\phi, p}^\Gamma(x, x) dx &= \left(2\pi h \int_0^{+\infty} r h(\ell(r)) dr \right) \log Y + 2\pi h \int_0^{+\infty} \log(r) r h(\ell(r)) dr \\ &\quad + \sum_{j=1}^h \kappa_{\Lambda_j} \text{vol} \Lambda_j \int_0^{+\infty} h(\ell(r)) dr \\ &\quad + \int_{D^Y} \left(\text{tr} k_{\phi, p}(x, x) + \sum_{\gamma \in \Gamma_{\text{lox}}} \text{tr}(\gamma^* k_{\phi, p}(x, \gamma x)) \right) dx + o(1). \end{aligned}$$

which finishes the proof. \square

Note that in $\kappa_j \text{vol} \Lambda_j$, only the term $\log \alpha(\Lambda_j)$ depends on the choice of the height function y_{P_j} .

4. TRACES IN BENJAMINI-SCHRAMM CONVERGENT SEQUENCES

In this section we fix a representation V (which we do not require to be strongly acyclic) and (except where indicated) fix a degree p and for $\phi \in \mathcal{S}(\mathbb{R})$ denote $k_\phi = k_{\phi, p}$.

4.1. Cusp-uniform sequences. If P is a parabolic subgroup and y_P is a height function at P then we may identify the unipotent radical N of P with the horosphere $y_P = 1$ and the conformal structure on N thus obtained does not depend on the chosen y_P . Since the uniformity of a set obviously only depends on the conformal structures of its elements we may define a cusp-uniform sequence as a sequence of lattices $\Gamma_n \subset G$ such that the set

$$\{(\Gamma_n)_P, n \geq 1, P \text{ is a } \Gamma\text{-rational parabolic}\}$$

is a uniform set of euclidean lattices. Every finite-volume manifold has a BS-convergent, cusp-uniform sequence of finite covers, in fact we have the following more precise result.

Proposition 4.1. *Let $\Gamma \subset G$ be a lattice, then there exists a cusp-uniform sequence $\Gamma_n \subset \Gamma$ which exhausts Γ ⁵.*

Proof. It is well-known that up to conjugation we may assume $\Gamma \subset \mathrm{SL}_2(F)$ for some number field F . As Γ is finitely generated there exists an $a \in \mathcal{O}_F$ such that $\Gamma \subset \mathrm{SL}_2(\mathcal{O}_F[a^{-1}])$. For an ideal $\mathfrak{J} \subset \mathcal{O}_F$ coprime to a we may define $\Gamma(\mathfrak{J})$ as the set of matrices in Γ congruent to the identity modulo \mathfrak{J} . Then the sequence of $\Gamma(n)$ for $n \in \mathbb{Z}$ coprime to a is clearly exhaustive and we claim that it is cusp-uniform. Indeed, if P is a Γ -rational parabolic we have $\Gamma_P = 1 + \mathbb{Z}X_1 + \mathbb{Z}X_2$ for some $X_1, X_2 \in M_2(\mathcal{O}_F[a^{-1}])$. Let \mathfrak{J} be the ideal in $\mathcal{O}_F[a^{-1}]$ generated by the entries of X_1 and X_2 and m the unique positive rational integer such that $\mathfrak{J} \cap \mathbb{Z} = m\mathbb{Z}$. Put $\Lambda_n = n\Gamma_P$; then the Λ_n are a uniform family of lattices in N and we have $\Lambda_n \subset \Gamma(n)_P \subset m^{-1}\Lambda_n$, so that $\{\Gamma(n)_P, n\}$ is uniform. Since the subgroups $\Gamma(n)$ are normal in Γ we need only consider a finite number of P and the claim follows. \square

4.2. Benjamini-Schramm convergence for manifolds with cusps. For an hyperbolic three-manifold $M = \Gamma \backslash \mathbb{H}^3$ and a point $x \in M$ let \tilde{x} be any lift of x to \mathbb{H}^3 and define

$$\ell_x = \min\{d(\tilde{x}, \gamma\tilde{x}), \gamma \in \Gamma, \gamma \neq 1_G\}.$$

Recall from [1] that a sequence $M_n = \Gamma_n \backslash \mathbb{H}^3$ is said to converge to \mathbb{H}^3 in the Benjamini-Schramm topology (hereafter abbreviated as M_n BS-converges to \mathbb{H}^3) if for any $R > 0$ we have

$$(4.1) \quad \frac{\mathrm{vol}\{x \in M_n : \ell_x \leq R\}}{\mathrm{vol} M_n} \xrightarrow{n \rightarrow \infty} 0.$$

Our two main sources of examples are exhaustive sequences (so that it follows from Proposition 4.1 that every hyperbolic three-manifold has a sequence of finite covers that is BS-convergent to \mathbb{H}^3 and cusp-uniform), and sequences of congruence lattices (see [1],[21]).

Now we fix a manifold $M = \Gamma \backslash \mathbb{H}^3$ and a set of Γ -invariant height functions. We recall that when dealing with a finite cover M' of M we will always consider the former as endowed with the height functions pulled back from M .

Lemma 4.2. *Let M_n be a sequence of finite-volume hyperbolic three-manifolds, h_n the number of cusps of M_n . If M_n BS-converges to \mathbb{H}^3 then $h_n = o(\mathrm{vol} M_n)$.*

Proof. Let ε be the Margulis constant for \mathbb{H}^3 . Then there are h_n noncompact components in the ε -thin part of M_n , which we denote by E_1, \dots, E_{h_n} . For each j we have that $\mathrm{vol} E_j \geq c\varepsilon^2$ where c is absolute, and it follows that for any $R \geq \varepsilon$ we have

$$\mathrm{vol}\{x \in M_n : \ell_x \leq R\} \geq c'h_n$$

so that (4.1) implies that $h_n = o(\mathrm{vol} M_n)$. \square

⁵i.e. any $\gamma \in \Gamma$ belongs to at most a finite number of the Γ_n

For a lattice Γ in G let Γ_{lox} be the set of loxodromic elements in Γ and for any $x \in \mathbb{H}^3$ and $r > 0$ put

$$\mathcal{N}_\Gamma(x, r) = |\{\gamma \in \Gamma_{\text{lox}}, d(x, \gamma x) \leq r\}|.$$

We have the following criterion for a sequence of noncompact hyperbolic three-manifolds to BS-converge.

Proposition 4.3. *Let M_n be a cusp-uniform sequence of finite covers of a hyperbolic three-manifold M and h_n the number of cusps of M_n . Then M_n BS-converges to \mathbb{H}^3 if and only if we have for all $r > 0$:*

$$(4.2) \quad \frac{\int_{M_n} \mathcal{N}_{\Gamma_n}(x, r) dx}{\text{vol } M_n} \xrightarrow{n \rightarrow \infty} 0.$$

Proof. Suppose that M_n BS-converges to \mathbb{H}^3 . We need to show that for any $r > 0$ the integral of $\mathcal{N}_{\Gamma_n}(\cdot, r)$ on $(M_n)_{\leq r} = \{x \in M_n, \inf_x(M_n) \leq r\}$ goes to 0 as $n \rightarrow \infty$. A well-known argument shows that there are constants C, c (depending on the injectivity radius of M) such that $\mathcal{N}_{\Gamma_n}(x, r) \leq Ce^{cr}$ so that

$$\int_{(M_n)_{\leq r}} \mathcal{N}_{\Gamma_n}(x, r) dx \leq Ce^{cr} \text{vol}(M_n)_{\leq r}$$

and (4.2) immediately follows from the definition (4.1).

To prove the converse it is enough to show that the volume of the part of M_n where there is a unipotent element with displacement less than r is small, that is if we put

$$(M_n)_{U, \leq r} = \{x \in M_n : \exists P, \tilde{x}, \eta \in (\Gamma_n)_P - \{1_G\} \text{ such that } d(x, \eta x) \leq r\}$$

we have $\text{vol}(M_n)_{U, \leq r} = o(\text{vol } M_n)$. By Lemma 2.1 below there is a $c > 0$ (depending on Γ and on the choice of the Γ -invariant height functions) such that $(M_n)_{U, \leq r}$ is contained in $\bigcup_j \{y_j \leq cr^{-1}\}$. It follows that

$$\begin{aligned} \text{vol}(M_n)_{U, \leq r} &\leq \sum_{j=1}^{h_n} \text{vol}\{y_j \geq cr^{-1}\} \\ &\ll \sum_{j=1}^{h_n} \int_{cr^{-1}}^{+\infty} \frac{dy}{y^3} = r^2 h_n \end{aligned}$$

where the constant on the second line depends only on Γ . Thus the BS-convergence follows from the fact that $h_n = o(\text{vol } M_n)$ and for the proof of this we refer to [22, Proposition 4.7] □

4.3. Convergence of traces. Let D be a fundamental domain for Γ acting on \mathbb{H}^3 ; the normalized trace of k_ϕ with respect to Γ is defined to be

$$(4.3) \quad \text{Tr}_\Gamma k_\phi = \int_D \text{tr } k_\phi(x, x) dx.$$

Since k_ϕ is G -invariant (cf. (3.7)) we get that $\text{Tr}_\Gamma k_{\phi, p} = \text{vol}(D) \text{tr } k_{\phi, p}(x_0, x_0)$ for any $x_0 \in \mathbb{H}^3$.

Theorem 4.4. *Let Γ be a lattice in $\text{SL}_2(\mathbb{C})$ and Γ_n a cusp-uniform sequence of finite-index subgroups in Γ which BS-converges to \mathbb{H}^3 . Then we have the limit*

$$(4.4) \quad \lim_{n \rightarrow \infty} \frac{\text{Tr}_R K_\phi^{\Gamma_n}}{[\Gamma : \Gamma_n]} = \text{Tr}_\Gamma k_\phi$$

Proof. We let D_n be a fundamental domain in \mathbb{H}^3 for Γ_n and h_n be the number of cusps of M_n . We choose representatives P_1, \dots, P_{h_n} of the Γ_n -classes of Γ -rational parabolic subgroups and denote by $\Lambda_{n,j}$ the lattice $(\Gamma_n)_{P_j}$ inside N_j (where N_j is the unipotent radical of P_j , identified with the horosphere $\{y_{P_j} = 1\}$).

For $p = 0, 1, 2, 3$ we have $3d \geq \dim V \otimes \wedge^p \mathfrak{P}$ so that

$$\mathrm{tr}(\gamma^* k_\phi(x, \gamma x)) \leq 3d |\rho(\gamma^{-1})|_V |k_\phi(x, \gamma x)|.$$

For $x = gK, y = hK \in \mathbb{H}^3$ we put $H(d(x, y)) = 3d |\rho(g^{-1}h)| |k_\phi(x, y)|$, then we have $H(r) \ll e^{-Ar}$ for all $A > 0$ as $r \rightarrow \infty$. We first want to estimate:

$$H_n = \int_{D_n} \sum_{\gamma \in \Gamma_{\mathrm{lox}}} H(d(x, \gamma x)) dx.$$

which is done in the following lemma.

Lemma 4.5. *If M_n BS-converges to \mathbb{H}^3 then $H_n = o(\mathrm{vol} M_n)$.*

Proof. This is dealt with exactly as in [1]. We have

$$\int_{D_n^Y} H(x) dx = \int_{D_n^Y} \int_0^{+\infty} \frac{dh}{dr} \mathcal{N}_{\Gamma_n}(x, r) dr dx \leq \int_0^{+\infty} \frac{dh}{dr} \int_{D_n^Y} \mathcal{N}_{\Gamma_n}(x, r) dx dr$$

and since $\mathcal{N}_{\Gamma_n}(x, r) \leq Ce^{cr}$ for absolute C, c the result follows from (4.3) and Lebesgue's dominated convergence theorem. \square

The summand $2\pi h_n \int_0^{+\infty} r \log(r) h(\ell(r)) dr$ is an $o(\mathrm{vol} M_n)$ by Lemma 4.2. It remains to see that we have

$$U_n := \sum_{j=1}^{h_n} \kappa_{j,n} \mathrm{vol}(\Lambda_{n,j}) \int_0^{+\infty} r h(\ell(r)) dr = o(\mathrm{vol} M_n).$$

We recall that $\kappa_{j,n} = \int_{\alpha_n^j}^{+\infty} E_{\Lambda_{n,j}}(\rho) \frac{d\rho}{\rho^3} + \frac{\pi(1+2\log \alpha_n^j)}{\mathrm{vol} \Lambda_{j,n}}$. Since by the hypotheses of the theorem the $\Lambda_{n,j}$ are a uniform family we get from Lemma 2.3 the following estimate:

$$\int_{\alpha_n^j}^{+\infty} E_{\Lambda_{n,j}}(\rho) \frac{d\rho}{\rho^3} \ll \frac{1}{\alpha_n^j} \int_{\alpha_n^j}^{+\infty} \frac{d\rho}{\rho^2} \ll (\alpha_n^j)^{-2} \ll (\mathrm{vol} \Lambda_{n,j})^{-1}$$

with a constant that does not depend on n or j , and it follows that

$$\kappa_{j,n} \mathrm{vol} \Lambda_{j,n} \ll 1 + \log \alpha_n^j.$$

so that we get $U_n \ll \sum_{j=1}^{h_n} \log \alpha_n^j$. To conclude that the right-hand side is an $o(\mathrm{vol} M_n)$ (which finishes the proof of the Theorem) we use the following stronger result.

Lemma 4.6. *Under the assumptions of the theorem we have $\sum_{j=1}^{h_n} \alpha_n^j = o(\mathrm{vol} M_n)$.*

Proof. We show that for all $C > 0$ we have

$$(4.5) \quad \limsup_n \frac{\sum_{j=1}^{h_n} \alpha_n^j}{\mathrm{vol} M_n} \leq C^{-1}.$$

We order the P_j so that α_n^j be increasing with j and denote by h_n^C the largest index such that $\alpha_n^j < C$ for all $j \leq h_n^C$. Then:

$$\begin{aligned} \sum_{j=1}^{h_n} \alpha_n^j &\ll Ch_n^C + \sum_{j=h_n^C+1}^{h_n} (\alpha_n^j)^{-1} [\Lambda_j : \Lambda_{n,j}] \ll Ch_n + (\alpha_n^{h_n^C+1})^{-1} \sum_{j=h_n^C+1}^{h_n} [\Lambda_j : \Lambda_{n,j}] \\ &\leq Ch_n + C^{-1} h_1 [\Gamma : \Gamma_n] \end{aligned}$$

where h_1 is the number of cusps of M . The conclusion (4.5) then follows at once from Lemma 4.2. \square

\square

5. APPROXIMATION FOR ANALYTIC TORSIONS

From now on we fix a strongly acyclic representation ρ, V of G and all forms are taken with coefficients in E_ρ . We define L^2 - and regularized torsions below, and then prove the following result.

Theorem 5.1. *Let Γ_n be a cusp-uniform sequence of finite-index subgroups of a lattice $\Gamma \subset G$ such that the sequence $M_n := \Gamma_n \backslash \mathbb{H}^3$ BS-converges to \mathbb{H}^3 . Suppose in addition that there exists $\varepsilon > 0$ and a sequence $a_n = o(\text{vol } M_n)$ such that for all $n \geq 1$ and $u \in [-\varepsilon, \varepsilon]$, l we have*

$$\text{tr} \left(\Psi_l(iu)^{-1} \frac{d\Psi_l(iu)}{du} \right) \leq a_n, \quad \text{tr} \left(\Phi_l(iu)^{-1} \frac{d\Phi_l(iu)}{du} \right) \leq a_n.$$

Then we have

$$(5.1) \quad \lim_{n \rightarrow \infty} \frac{T_R(M_n; V)}{\text{vol } M_n} = t^{(2)}(V).$$

5.1. Heat kernels. For $\phi(u) = e^{-tu}$ the kernel $k_{\phi,p}$ (resp. $K_{\phi,p}^\Gamma$) is called the heat kernel of \mathbb{H}^3 (resp. of M). We will use the bounds for the heat kernel given by the following theorem which are well-known (see [22, Section 0.4]).

Theorem 5.2. *Let X be a Riemannian manifold (which may have boundary) of dimension n whose curvature is bounded above by C and injectivity radius is bounded below by δ . Let E be an euclidean vector bundle on X and Δ^p the Laplacian on p -forms with coefficients in E (with absolute boundary conditions if need be). Let $t_0 > 0$; there exists constants C_1 (depending on t_0, n, δ, E and C) and C_2 (depending only on n) such that for all $x, y \in X$ and $t \in (0, t_0)$ we have*

$$|e^{-t\Delta^p}(x, y)| \leq C_1 t^{-d/2} e^{-\frac{d(x,y)^2}{C_2 t}}.$$

We will also make use of the following fact about the heat kernel.

Theorem 5.3. *There exists $\alpha_k^p \in C^\infty(G, \text{End}(\wedge^p \mathfrak{p} \otimes V))$ such that for all $x \in \mathbb{H}^3$ we have the asymptotic expansion at $t \rightarrow 0$*

$$g^* e^{-t\Delta^p[\mathbb{H}^3]}(x, gx) = \sum_{k=-3}^m \alpha_k^p(g) e^{-\frac{d(x,gx)^2}{C_2 t}} t^{\frac{k}{2}} + O(t^{\frac{m+1}{2}}).$$

(Note that only the germ at 0 of the α_k^p is well-defined)

Proof. Let $W = V \otimes \wedge^p \mathfrak{p}^*$ with its right K -action σ , Ω_K the Casimir of K and p_t the heat kernel (for functions) of G with the bi- K -invariant, left- G -invariant metric. Then $\sigma(\Omega_K)$ is positive definite and letting $O \in \mathbb{H}^3$ be the fixed point of K we have (see for example [3, (3.8.1)]):

$$e^{-t\Delta^p[\mathbb{H}^3]}(O, gO) = \int_{K \times K} p_t(k_1^{-1} g k_2^{-1}) \sigma(k_1 e^{2t\Omega_K} k_2) dk_1 dk_2.$$

Thus the wanted expansion follows from that of p_t which is stated for example in [8, p. 272] (cf. [22, Section 0.4.2]). \square

5.2. Asymptotic expansion of the heat kernel at $t \rightarrow 0$. We will need the following result to define the regularized analytic torsion (see also [16, Proposition 6.9]).

Proposition 5.4. *For all $p = 0, 1, 2, 3$ and $m \geq 1$ there are coefficients $a_0^p, \dots, a_m^p, b_0^p, \dots, b_m^p$ and a function H^p such that*

$$(5.2) \quad \text{Tr}_R(e^{-t\Delta^p[M]}) = \sum_{k=0}^{m+3} \left(a_k^p t^{\frac{k-3}{2}} + b_k^p t^{\frac{k-1}{2}} \log t \right) + H^p(t)$$

and $H^p(t) \ll t^{\frac{m+1}{2}}$ as $t \rightarrow 0$.

Proof. We fix p and put $h_t^p(\ell) = \text{tr}(n^* e^{-t\Delta^p[M]}(x, nx))$ for a unipotent element $n \in G$ and a $x \in \mathbb{H}^3$ such that $d(x, nx) = \ell$. We choose a fundamental domain D for Γ and define

$$\begin{aligned} S_1(t) &= 2\pi h \int_0^{+\infty} r \log(r) h_t^p(\ell(r)) dr, \\ S_2(t) &= \sum_{j=1}^h \kappa_j \text{vol } \Lambda_j \int_0^{+\infty} r h_t^p(\ell(r)) dr \\ S_3(t) &= \int_D \sum_{\gamma \in \Gamma_{\text{lox}}} \text{tr}(\gamma^* e^{-t\Delta^p[M]}(x, \gamma x)) \end{aligned}$$

so that by Proposition 3.4 we have:

$$(5.3) \quad \text{Tr}_R e^{-t\Delta^p[M]} = \text{Tr}_\Gamma e^{-t\Delta^p[\mathbb{H}^3]} + S_1(t) + S_2(t) + S_3(t).$$

Putting $g = 1_G$ in Theorem 5.3 and integrating over D we get an expansion

$$(5.4) \quad \text{Tr}_\Gamma e^{-t\Delta^p[\mathbb{H}^3]} = \text{vol } D \sum_{k=0}^{m+3} f_k^p t^{-\frac{k}{2}}$$

where the f_k^p are absolute coefficients, which takes care of the first summand.

Now we deal with S_3 ; if we put

$$\ell_0 = \inf\{d(x, \gamma x), x \in \mathbb{H}^3, \gamma \in \Gamma_{\text{lox}}\}$$

we get:

$$\begin{aligned} \sum_{\gamma \in \Gamma_{\text{lox}}} \text{tr}(\gamma^* e^{-t\Delta^p[M]}(x, \gamma x)) &\leq \sum_{\gamma \in \Gamma_{\text{lox}}} C_1 t^{-\frac{3}{2}} e^{-\frac{d(x, \gamma x)^2}{C_2 t}} \\ (5.5) \quad &= C_1 \int_{\ell_0}^{+\infty} e^{-\frac{\ell^2}{C_2 t}} d\mathcal{N}_\Gamma(x, \ell) \\ &\ll \int_{\ell_0}^{+\infty} t^{-\frac{5}{2}} \ell e^{-\frac{\ell^2}{C_2 t}} \mathcal{N}_\Gamma(x, \ell) d\ell \\ &\ll t^{-\frac{5}{2}} e^{-\frac{\ell_0^2}{C_2 t}} \end{aligned}$$

so that $S_3(t)$ is actually an $o(t^{\frac{m+1}{2}})$ for all $m > 0$.

To deal with S_1 and S_2 we will use the following expansion at $t \rightarrow 0$, which follows immediately from Theorem 5.3:

$$(5.6) \quad h_t^p(\ell) = \sum_{k=-3}^m b_k^p(\ell) e^{-\frac{\ell^2}{C_2 t} t^{\frac{k}{2}}} + O(t^{\frac{m+1}{2}})$$

Let ω be smooth in a neighbourhood of $[0, 1]$, we show that for every integer $m > 1$ there are constants c_l, c'_l $l = 1, \dots, m+1$ such that

$$(5.7) \quad \int_0^1 r \log(r) \omega(r) e^{-\ell(r)^2/C_2 t} dr - \sum_{l=2}^m t^{l/2} (c_l + c'_l \log t) \leq c_{m+1} t^{(m+1)/2}$$

Since $\ell \mapsto r \log(r)/\ell \log(\ell)$ is a smooth function near 0 we need only show that for a smooth function ω_0 on $[0, 1]$ there an expansion of the right form as $t \rightarrow 0$ of:

$$\int_0^1 \ell \log(\ell) \omega_0(\ell) e^{-\frac{\ell^2}{t}} d\ell = t \log t \int_0^{t^{-1/2}} \ell \omega_0(t^{\frac{1}{2}} \ell) e^{-\ell^2} d\ell + t \int_0^{t^{-1/2}} \ell \log(\ell) \omega_0(t^{\frac{1}{2}} \ell) e^{-\ell^2} d\ell$$

This is an immediate consequence of Taylor's formula applied to ω_0 at 0 and of the following easy estimate:

$$\int_0^{t^{-\frac{1}{2}}} \ell^k e^{-\ell^2} d\ell = \int_0^{+\infty} \ell^k e^{-\ell^2} d\ell + O(t^{-\frac{k}{2}} e^{-\frac{1}{t}}).$$

We get an expansion similar to (5.7) (but without the $\log t$ factor) for $\int_0^1 r \omega(r) e^{-\frac{\ell(r)^2}{C_2 t}} dr$ using the same argument. We finally get that for all $m \geq 1$ there are coefficients c_k^p, d_k^p, e_k^p such that we have at $t \rightarrow 0$

$$(5.8) \quad \begin{aligned} \int_0^{+\infty} r \log(r) h_t^p(\ell(r)) dr &= \sum_{k=-3}^m c_k^p t^{\frac{k}{2}} + \sum_{k=-1}^m e_k^p t^{\frac{k}{2}} \log t + O(t^{\frac{m+1}{2}}), \\ \int_0^{+\infty} r h_t^p(\ell(r)) dr &= \sum_{k=-3}^m e_k^p t^{\frac{k}{2}} + O(t^{\frac{m+1}{2}}) \end{aligned}$$

and the expansion for the regularized trace thus follows from this, (5.3), (5.4) and (5.5). \square

5.3. Definition of analytic torsions.

5.3.1. Regularized torsion. We fix a nonuniform torsion-free lattice Γ in G . As usual we also denote by Γ Euler's Gamma function defined for $\text{Re}(s) > 0$ by the formula $\Gamma(s) = \int_0^{+\infty} e^{-t} t^{s-1} \frac{dt}{t}$ and meromorphically continued to \mathbb{C} . It has a simple pole at each $s = -n$ for $n \in \mathbb{N}$ and no zeroes, so that $1/\Gamma$ is holomorphic on \mathbb{C} . We want to check that

$$(5.9) \quad \zeta_p(s) := \frac{1}{\Gamma(s)} \int_0^{+\infty} \text{Tr}_R(e^{-t\Delta^p[M]}) t^s \frac{dt}{t}$$

defines a holomorphic function on the half-plane $\text{Re}(s) > 3/2$ which can be continued to a meromorphic function on \mathbb{C} which is holomorphic at 0. The large-time convergence of the integral is ensured by the spectral gap for the Laplacian⁶ as follows. The spectral expansion (3.15) applied to the heat kernel yields, for example for $p = 1$, the following estimate as $t \rightarrow \infty$:

$$\text{Tr}_R e^{-t\Delta^1[M]} = \text{Tr}(e^{-t\Delta^1[M]})_{\text{disc}} + e^{-t\lambda_V} \text{Truc}, \quad 7$$

whence it follows that $\text{Tr}_R e^{-t\Delta^1[M]} \ll e^{-\lambda_0 t}$ as $t \rightarrow \infty$, where $\lambda_0 = \min(\lambda_V, \lambda_{\text{cusp}}^1)$ and $\lambda_{\text{cusp}}^1 > 0$ is the smallest eigenvalue of all $\Delta_{\text{cusp}}^1[M]$ (see Proposition 2.2). Thus we get that for any $t_0 > 0$ the integral $\int_{t_0}^{+\infty} \text{Tr}_R(e^{-t\Delta^p[M]}) t^s \frac{dt}{t}$ converges for all $s \in \mathbb{C}$. An easy computation moreover yields that

$$(5.10) \quad \frac{d}{ds} \left(\frac{1}{\Gamma(s)} \int_{t_0}^{+\infty} \text{Tr}_R(e^{-t\Delta^p[M]}) t^s \frac{dt}{t} \right)_{s=0} = \int_{t_0}^{+\infty} \text{Tr}_R(e^{-t\Delta^p[M]}) \frac{dt}{t}.$$

⁶When there's no spectral gap the integral converges only in a half-plane $\text{Re}(s) < c < 0$, see [19] or [16].

⁷ $\text{Truc} = \frac{1}{4} \sum_{l=-2q}^{2q} d_l e^{-t((1-\frac{|l|}{2})^2 + \lambda_V)} \text{tr} \Psi_l(0) - \frac{1}{2\pi} \int_{-\infty}^{+\infty} \sum_{l=-2q}^{2q} d_l e^{-t((1-\frac{|l|}{2})^2 + u^2 + \lambda_V)} \text{tr}(\Psi_l(iu)^{-1} \frac{d\Psi_l(iu)}{du}) du + \dots$

To deal with the small-time part we use the following classical lemma.

Lemma 5.5. *Let $\phi \in C^0(0, +\infty)$ such that there are integers $m, m' \geq 0$, coefficients a_k , $k = -m', \dots, m$ and a continuous function H so that*

$$\phi(t) =_{t \rightarrow 0} \sum_{k=-m'}^m a_k t^{-k/2} + H(t)$$

and $H(t) \ll t^{\frac{m+1}{2}}$ near 0. Then for all $t_0 > 0$ the integral $\frac{1}{\Gamma(s)} \int_0^{t_0} \phi(t) t^s \frac{dt}{t}$ converges on the half-plane $\operatorname{Re}(s) > m'/2$ and the holomorphic function it defines may be meromorphically continued to a function on $\operatorname{Re}(s) > 1/2$ which is regular at 0. Moreover we have the following equality ($\gamma := \Gamma'(1)$):

$$\frac{d}{ds} \left(\frac{1}{\Gamma(s)} \int_0^{t_0} \phi(t) t^s \frac{dt}{t} \right)_{s=0} = \sum_{k=1}^m \frac{2a_k t_0^{-k/2}}{k} - \gamma a_0 + \int_0^{t_0} H(t) \frac{dt}{t}.$$

Proof. For $\alpha \in \mathbb{C}$ the integral $\int_0^{t_0} t^{\alpha+s-1} dt$ converges absolutely on $\operatorname{Re}(s) > \alpha$ and defines a meromorphic function on \mathbb{C} with a single simple pole at $s = -\alpha$, and since $1/\Gamma$ has a zero at 0 and the integral $\int_0^{t_0} H(t) t^s dt/t$ converges for $\operatorname{Re}(s) > -m/2$ we get the first part. The formula for the derivative at 0 follows from a straightforward computation. \square

It follows from Proposition 5.4 and Lemma 5.5 that we may define the regularized determinant of the Laplacian by

$$(5.11) \quad \det_R \Delta^p[M] := \exp(\zeta'_p(0))$$

and the analytic torsion by

$$(5.12) \quad T_R(M) = \left(\prod_{p=0}^3 \det_R(\Delta^p[M])^{(-1)^p p} \right)^{\frac{1}{2}} = (\det_R(\Delta^0[M])^{-3} \det_R(\Delta^1[M]))^{\frac{1}{2}}$$

5.3.2. L^2 -torsion. The natural candidate to be the limit of finite torsions is the L^2 -torsion, cf. [14, Question 13.73]. The following definition does not depend on $t_0 > 0$:

$$(5.13) \quad \begin{aligned} \log T^{(2)}(M; V) &= \frac{1}{2} \sum_{p=1}^3 p(-1)^p \left(\frac{d}{ds} \left(\frac{1}{\Gamma(s)} \int_0^{t_0} \operatorname{Tr}_\Gamma(e^{-t\Delta^p[\mathbb{H}^3]}) t^{s-1} dt \right)_{s=0} \right. \\ &\quad \left. + \int_{t_0}^{+\infty} \operatorname{Tr}_\Gamma(e^{-t\Delta^p[\mathbb{H}^3]}) \frac{dt}{t} \right). \end{aligned}$$

The convergence of the first integral follows from the asymptotic expansion (5.4); the large-time convergence is obvious here because the Laplacian on \mathbb{H}^3 with coefficients in V has a spectral gap. We see that $\log T^{(2)}$ is a multiple of $\operatorname{vol} D$ and we denote by $t^{(2)}(V)$ the constant $\log(T^{(2)}(M, V))/\operatorname{vol}(M)$. This has been computed in all generality in [3] to yield (1.5)

5.4. Proof of Theorem 5.1.

5.4.1. Plan of proof. We naturally study small and large times separately. We want first to prove that for any $t_0 > 0$ the following limit holds:

$$(5.14) \quad \frac{1}{\operatorname{vol} M_n} \frac{d}{ds} \left(\int_0^{t_0} (\operatorname{Tr}_R(e^{-t\Delta^p[M_n]}) - \operatorname{Tr}_{\Gamma_n}(e^{-t\Delta^p[\mathbb{H}^3]})) t^s \frac{dt}{t} \right)_{s=0} \xrightarrow{n \rightarrow \infty} 0$$

The proof of this is more involved than that of the pointwise convergence of the traces since we have to control the asymptotics as $t \rightarrow 0$ of the heat kernels of M_n as $n \rightarrow \infty$. We are going to

go over the steps of the proof of Proposition 4.4 with extra care for the dependance in t of the majorations.

Once this is done we have to deal with the convergence of the large-time integral as n varies. What we need is the following limits, which we will prove right away in 5.4.2 below.

$$(5.15) \quad \lim_{t_0 \rightarrow +\infty} \left(\limsup_{n \rightarrow \infty} \int_{t_0}^{+\infty} \frac{\text{Tr}_R(e^{-t\Delta^p[M_n]})}{\text{vol } M_n} \frac{dt}{t} \right) = 0,$$

$$(5.16) \quad \lim_{t_0 \rightarrow \infty} \left(\int_{t_0}^{+\infty} \text{Tr}_\Gamma(e^{-t\Delta^p[\mathbb{H}^3]}) \frac{dt}{t} \right) = 0.$$

Assuming all these limits we can now conclude the proof of Theorem 5.1: the limit (5.14) above yields for all $t_0 > 0$

$$\limsup_{n \rightarrow \infty} \frac{\log T_R(M_n) - \log T^{(2)}(M_n)}{\text{vol } M_n} \leq \limsup_{n \rightarrow \infty} \left(\int_{t_0}^{+\infty} \frac{\text{Tr}_R(e^{-t\Delta^p[M_n]})}{\text{vol } M_n} \frac{dt}{t} \right) + \int_{t_0}^{+\infty} \text{Tr}_\Gamma(e^{-t\Delta^p[\mathbb{H}^3]}) \frac{dt}{t}$$

and by (5.15), (5.16) we get that the right-hand side goes to 0 as $t_0 \rightarrow +\infty$, so that the limit superior on the left must be 0.

5.4.2. Spectral gap and large times. Obviously (5.16) follows from the convergence of the integral. Now we prove (5.15) using the uniform spectral gap. Let $\lambda_0 > 0$ be the lower bound for the spectrum of all $\Delta^p[M_n]$, we have $\lambda_0 > 0$ by Proposition 2.2. Let $u \in [\varepsilon, +\infty)$, then the Maass-Selberg relations (3.5) for $Y = 1$ yield

$$-\langle \Psi(iu)^{-1} \frac{d\Psi(iu)}{du} \cdot v, v \rangle_{V_{\mathbb{C}}^h} = |T^1 E(iu, v)|_{L^2(M_n; V_{\mathbb{C}})}^2 + \frac{1}{iu} (\langle \Psi(iu) \cdot v, v \rangle_{V_{\mathbb{C}}^h} - \langle \Psi(-iu) \cdot v, v \rangle_{V_{\mathbb{C}}^h}).$$

As $\Psi(iu)$ is unitary the right-hand side is bounded below by $-2\varepsilon^{-1}$ and it follows that $2\varepsilon^{-1} 1_{V_{\mathbb{C}}^h} - \Psi(iu)^{-1} \frac{d\Psi(iu)}{du}$ is positive when $|u| \geq \varepsilon$. Using this, the fact that

$$\text{tr} \left(\frac{a_n}{h_n \dim V} 1_{V_{\mathbb{C}}^{h_n}} - \Psi_l(iu)^{-1} \frac{d\Psi_l(iu)}{du} \right) \geq 0$$

and the equality $e^{-\lambda t} = e^{-\lambda} e^{-\lambda(t-1)}$ for $t \geq 1$ we get

$$\begin{aligned} |\text{Tr}_R(e^{-t\Delta^0[M_n]})| &\leq e^{\lambda_0(t-1)} \text{Tr} e^{-t\Delta_{\text{disc}}^0[M_n]} + e^{-\lambda_V(t-1)} \left| \frac{1}{4} \sum_{l=-2q}^{2q} e^{-(\lambda_V + (1 - \frac{|l|}{2}))} \text{tr } \Psi_l(0) \right| \\ &\quad + e^{-\lambda_V(t-1)} \left| \sum_l \int_{-\infty}^{+\infty} e^{-(u^2 + (1 - |l|/2)^2)} \text{tr } \Psi_l(iu)^{-1} \frac{d\Psi_l(iu)}{du} du \right| + e^{-t\lambda_V} O(a_n + h_n) \\ &\leq e^{-\lambda_0(t-1)} \text{Tr}_R(e^{-\Delta^p[M_n]}) + e^{-t\lambda_V} o(\text{vol } M_n), \end{aligned}$$

whence it follows that

$$\sup_n \left(\frac{1}{\text{vol } M_n} \int_{t_0}^{+\infty} \text{Tr}_R(e^{-t\Delta^p[M_n]}) \frac{dt}{t} \right) \leq \int_{t_0}^{+\infty} e^{-\lambda_0(t-1)} \frac{dt}{t} \sup_n \text{Tr}_R(e^{-\Delta^p[M_n]}) + C e^{-\lambda_V t_0}.$$

By Theorem 4.4 the sequence $\text{Tr}_R(e^{-\Delta^p[M_n]})$ is bounded, so that the right-hand side above is bounded in n and goes uniformly to 0 as $t_0 \rightarrow \infty$.

5.4.3. *Small times.* To deal with the small-time part we analyse each of the terms in (5.3). Recall that for $j = 1, \dots, h_n$ we have put $\Lambda_{j,n} = (\Gamma_{P_j})_n$, $\alpha_n^j = \alpha(\Lambda_{n,j})$ and

$$\kappa_{j,n} = \int_{\alpha_n^j}^{+\infty} E_{\Lambda_{j,n}}(\rho) \frac{d\rho}{\rho^3} - \frac{\pi(1 + 2 \log \alpha_n^j)}{\text{vol } \Lambda_{j,n}}.$$

Then we get that

$$\begin{aligned} \text{Tr}_R e^{-t\Delta^p[M_n]} - \text{Tr}_\Gamma e^{-t\Delta^p[\mathbb{H}^3]} &= \int_{D_n} \sum_{\gamma \in \Gamma_{\text{lox}}} \text{tr } \gamma^* e^{-t\Delta^p[\mathbb{H}^3]}(x, \gamma x) dx \\ &\quad + 2\pi h_n \int_0^{+\infty} r \log(r) h_t^p(\ell(r)) dr + \sum_{j=1}^{h_n} \kappa_{n,j} \text{vol } \Lambda_{n,j} \int_0^{+\infty} r h_t^p(\ell(r)) dr \\ &=: T_1 + T_2. \end{aligned}$$

By the estimates from Theorem 5.2 we have

$$\begin{aligned} \frac{d}{ds} \left(\int_0^{t_0} T_1 t^s \frac{dt}{t} \right)_{s=0} &\ll \int_0^{t_0} \int_{D_n} \sum_{\gamma \in \Gamma_{\text{lox}}} e^{-\frac{d(x, \gamma x)^2}{C_2 t}} dx t^{-\frac{5}{2}} \frac{dt}{t} \\ &= \int_{D_n} \sum_{\gamma \in \Gamma_{\text{lox}}} \left(\int_0^{t_0} e^{-\frac{d(x, \gamma x)^2}{C_2 t}} t^{-\frac{5}{2}} \frac{dt}{t} \right) dx \end{aligned}$$

and the right-hand side is an $o(\text{vol } M_n)$ by Lemma 4.5.

Now we deal with S_1 and S_2 ; put:

$$\begin{aligned} \Xi(s) &= \int_0^{t_0} \int_0^{+\infty} r \log(r) h_t^p(\ell(r)) dr t^s \frac{dt}{t}, \\ \Theta(s) &= \int_0^{t_0} \int_0^{+\infty} r h_t^p(\ell(r)) dr t^s \frac{dt}{t}. \end{aligned} \tag{5.17}$$

It follows from (5.8) and Lemma 5.5 that Ξ, Θ extend to meromorphic functions on \mathbb{C} which are holomorphic at 0 and we get

$$\frac{d}{ds} \left(\int_0^{t_0} T_2 t^s \frac{dt}{t} \right)_{s=0} = 2\pi h_n \frac{d\Xi}{ds}(0) + \left(\sum_{j=1}^{h_n} \kappa_{n,j} \text{vol } \Lambda_{n,j} \right) \frac{d\Theta}{ds}(0).$$

On the other hand we have seen that $\sum_{j=1}^{h_n} \kappa_{n,j} \text{vol } \Lambda_{n,j} = o(\text{vol } M_n)$ in the proof of Theorem 4.4, and $h_n = o(\text{vol } M_n)$ by Lemma 4.2, so that the right-hand side itself is $o(\text{vol } M_n)$, which concludes the proof.

6. ASYMPTOTIC EQUALITY BETWEEN REGULARIZED AND ABSOLUTE TORSIONS

We prove here the following theorem, which follows from Proposition 6.7 (which deals with the large-time part of the proof) and Corollary 6.6 (the small-time bit) in the same way as in the proof of Theorem 5.1.

Theorem 6.1. *Let V be a strongly acyclic representation of G . Let Γ be a nonuniform lattice in G and Γ_n a cusp-uniform sequence of finite-index subgroups in Γ such that the manifolds $M_n = \Gamma_n \backslash \mathbb{H}^3$ BS-converge to \mathbb{H}^3 . There exists a sequence Y^n such that the following limit holds.*

$$(6.1) \quad \frac{\log T_R(M_n; V) - \log T_{\text{abs}}(M_n^{Y^n}; V)}{\text{vol } M_n} \xrightarrow{n \rightarrow \infty} 0.$$

We deduce from the theorem above an asymptotic equality between regularized torsion and a *combinatorial* absolute torsion.

Theorem 6.2. *Notations as above, suppose that Γ preserves a lattice $V_{\mathbb{Z}} \subset V$ and let τ_{abs} be defined by (6.3) below. Then we have the limit*

$$(6.2) \quad \frac{\log T_R(M_n; V) - \log \tau_{\text{abs}}(M_n^{Y^n}; V_{\mathbb{Z}})}{\text{vol } M_n} \xrightarrow{n \rightarrow \infty} 0.$$

Besides Theorem 6.1, the main ingredient used in the proof is the Cheeger-Müller theorem for manifolds with boundary proven in all generality by Brüning and Ma.

6.1. Absolute torsions.

6.1.1. *Generalities.* Let X be a compact Riemannian manifold with boundary and V a real flat vector bundle on X with a Euclidean metric. Then the space $\Omega^p(M; V)$ of smooth p -forms on X with coefficients in V is operated upon by the Hodge Laplacian $\Delta^p[X]$. The restriction of $\Delta^p[X]$ to the forms satisfying absolute conditions on the boundary (i.e. the boundary restrictions of $*f$ and $*df$ are zero, where $*$ is the Hodge star) admits an essentially autoadjoint extension Δ_{abs} to the space $L^2\Omega^p(X; V)$ of square-integrable p -forms. We thus may form the associated heat kernel $e^{-t\Delta_{\text{abs}}^p[X]}$ which is the convolution by a smooth kernel $e^{-t\Delta_{\text{abs}}^p[X]}(.,.)$, is trace-class and has an asymptotic expansion $\text{Tr}(e^{-t\Delta_{\text{abs}}^p[X]}) = a_3 t^{-\frac{3}{2}} + \dots + a_0 + O(t^{\frac{1}{2}})$ as $t \rightarrow 0$ (cf. [10, Theorem 1.11.4]). On the other hand the spectrum of $\Delta_{\text{abs}}^p[X]$ is discrete and thus we have an estimate

$$\text{Tr}(e^{-t\Delta_{\text{abs}}^p[X]}) - \dim \ker(\Delta_{\text{abs}}^p[X]) \ll e^{-\lambda_1 t}$$

where λ_1 is its smallest positive eigenvalue. Thus the zeta function

$$\zeta_{p,\text{abs}}(s) := \frac{1}{\Gamma(s)} \int_0^{+\infty} (\text{Tr}(e^{-t\Delta_{\text{abs}}^p[X]}) - \dim \ker(\Delta_{\text{abs}}^p[X])) t^s \frac{dt}{t}$$

is well-defined for $\text{Re}(s) > 3/2$ and may be continued to a meromorphic function on \mathbb{C} which is holomorphic at 0. One then defines $\det(\Delta_{\text{abs}}^p[X]) = \exp(\zeta'_{p,\text{abs}}(0))$ and

$$T_{\text{abs}}(X; V) = \left(\prod_{p=1}^{\dim X} \det(\Delta_{\text{abs}}^p[X])^{(-1)^p p} \right)^{\frac{1}{2}}.$$

Now suppose that $\pi_1(X)$ preserves a lattice $V_{\mathbb{Z}}$ in V ; we can then define the integral homology $H_*(X; V_{\mathbb{Z}})$ and we have a decomposition $H_p(X; V_{\mathbb{Z}}) = H_p(X; V_{\mathbb{Z}})_{\text{libre}} \oplus H_p(X; V_{\mathbb{Z}})_{\text{tors}}$. The free part $H_p(X; V_{\mathbb{Z}})_{\text{libre}}$ is a lattice in $\ker(\Delta_{\text{abs}}^p[X])$ so we can define its covolume $\text{vol } H_p(X; V_{\mathbb{Z}})_{\text{libre}}$. We put:

$$(6.3) \quad \tau_{\text{abs}}(X; V_{\mathbb{Z}}) = \prod_{p=0}^{\dim X} \left(\frac{|H_p(X; V_{\mathbb{Z}})_{\text{tors}}|}{\text{vol } H_p(X; V_{\mathbb{Z}})_{\text{libre}}} \right)^{(-1)^p}.$$

6.1.2. *Truncated hyperbolic manifolds.* Let $M = \Gamma \backslash \mathbb{H}^3$ be a complete hyperbolic manifold with cusps y_1, \dots, y_h a set of Γ -invariant height functions. Then for $Y \in [1, +\infty)^h$ the set

$$\widetilde{M}^Y = \{x \in \mathbb{H}^3, \forall j = 1, \dots, h : y_j(x) \leq Y_j\}$$

is the universal cover of M^Y and we have the following series expansion for the heat kernel

$$(6.4) \quad e^{-t\Delta_{\text{abs}}^p[M^Y]}(x, y) = \sum_{\gamma \in \Gamma} \gamma^* e^{-t\Delta_{\text{abs}}^p[\widetilde{M}^Y]}(x, \gamma y).$$

6.2. Small-time asymptotics.

6.2.1. Small-time behaviour of truncated integrals.

Proposition 6.3. *We put*

$$S(n, t, Y) = \int_{D_n^Y} \sum_{\gamma \in \Gamma_n, \gamma \neq 1} e^{-d(x, \gamma x)^2 / C_2 t}$$

then for all $t_0 > 0$ the function $s \mapsto \int_0^{t_0} S(n, t, Y) t^{s-\frac{3}{2}} dt / t$ is holomorphic on \mathbb{C} and there is a sequence $Y^n \in [1, +\infty)^{h_n}$ such that $\min_{j=1, \dots, h_n} (Y_j / \alpha_n^j) \xrightarrow{n \rightarrow \infty} 0$ and we have:

$$\frac{d}{ds} \left(\frac{1}{\Gamma(s)} \int_0^{t_0} \frac{S(n, t, Y^n)}{\text{vol } M_n} t^{s-\frac{3}{2}} \frac{dt}{t} \right)_{s=0} = \int_0^{t_0} t^{-\frac{3}{2}} \frac{S(n, t, Y^n)}{\text{vol } M_n} \frac{dt}{t} \xrightarrow{n \rightarrow \infty} 0.$$

Proof. We write

$$H_t(x) = \sum_{\gamma \in (\Gamma_n)_{\text{lox}}} t^{-\frac{3}{2}} e^{-\frac{d(x, \gamma x)^2}{C_2 t}}, \quad U_t(x) = \sum_{j=1}^{h_n} \sum_{\gamma \in \Gamma_n / (\Gamma_n)_{P_j}} \sum_{\eta \in \Lambda_{n,j} - \{0\}} t^{-\frac{3}{2}} e^{-\frac{d(x, \gamma \eta \gamma^{-1} x)^2}{C_2 t}}.$$

Let $\ell = \min_{\gamma \in \Gamma_{\text{lox}}} \min_{x \in \mathbb{H}^3} d(x, \gamma x)$; as in the proof of 4.5 we get

$$\int_0^{t_0} \int_{D_n^{Y^n}} H_t(x) dx \frac{dt}{t} \leq \int_0^{t_0} \int_{\ell}^{+\infty} 2t^{-\frac{5}{2}} r e^{-\frac{r^2}{C_2 t}} \int_{D_n} \mathcal{N}_{\Gamma_n}(x, r) dx dr \frac{dt}{t}$$

and the same argument used there yields that $\int_0^{t_0} \int_{D_n^{Y^n}} H_t(x) dx dt / t = o(\text{vol } M_n)$.

To deal with U_t we use less refined majorations than those given by (3.18). For all $n \geq 1, j = 1, \dots, h_n$ and $Y_j > 0$ we have

$$\sum_{\gamma \in \Gamma_n / (\Gamma_n)_{P_j}} \sum_{\eta \in \Lambda_{n,j} - \{0\}} t^{-\frac{3}{2}} e^{-\frac{d(x, \gamma \eta \gamma^{-1} x)^2}{C_2 t}} = \int_{B_{n,j}^{Y_j}} \sum_{\eta \in \Lambda_{n,j} - \{0\}} e^{-\frac{d(x, \eta x)^2}{C_2 t}} dx$$

(recall that $B_{n,j}^Y = B_{\Lambda_{n,j}}^Y$ was defined by (3.17)). We get:

$$\begin{aligned} \int_{D_n^Y} \sum_{\substack{\gamma \in \Gamma_n / (\Gamma_n)_{P_j} \\ \eta \in \Lambda_{n,j} - \{0\}}} t^{-\frac{3}{2}} e^{-\frac{d(x, \gamma \eta \gamma^{-1} x)^2}{C_2 t}} &= \text{vol } \Lambda_{n,j} \int_0^{Y_j} \int_{\alpha_n^j}^{+\infty} e^{-\frac{\ell(r/y)^2}{C_2 t}} d\mathcal{N}_{n,j}^*(r) \frac{dy}{y^3} \\ &= \text{vol } \Lambda_{n,j} \int_0^{Y_j} \int_{\alpha_n^j}^{+\infty} \frac{1}{C_2 t} \frac{1}{y} \frac{d\ell}{dr}(r/y) e^{-\frac{\ell(r/y)^2}{C_2 t}} \mathcal{N}_{n,j}^*(r) dr \frac{dy}{y^3} \\ &= \text{vol } \Lambda_{n,j} \int_0^{Y_j} \int_{\alpha_n^j/y}^{+\infty} \frac{1}{C_2 t} \frac{d\ell}{dr} e^{-\frac{\ell(r)^2}{C_2 t}} \mathcal{N}_{n,j}^*(ry) dr \frac{dy}{y^3}. \end{aligned}$$

By Lemma 2.3 and the uniformity of the $\Lambda_{n,j}$ we get $\mathcal{N}_{n,j}^*(ry) \ll (ry)^2 / \text{vol } \Lambda_{n,j}$ where the constant does not depend on n, j . It follows that

$$(6.5) \quad \int_{D_n^Y} \sum_{\substack{\gamma \in \Gamma_n / (\Gamma_n)_{P_j} \\ \eta \in \Lambda_{n,j} - \{0\}}} t^{-\frac{3}{2}} e^{-\frac{d(x, \gamma \eta \gamma^{-1} x)^2}{C_2 t}} dx \ll \int_0^{Y_j} \int_{\alpha_n^j/y}^{+\infty} r^2 e^{-\frac{\ell(r)^2}{C_2 t}} dr \frac{dy}{y} t^{-\frac{5}{2}}.$$

Now we split the integration between $r \geq 2$ and $\alpha_n^j/y \leq r \leq 2$. When $r \geq 2$ it follows from Lemma 2.1 that $\ell(r) \gg \log r$ and we obtain:

$$(6.6) \quad \int_0^{t_0} \int_0^{Y_j} \int_2^{+\infty} r^2 e^{-\frac{\ell(r)^2}{C_2 t}} dr \frac{dy}{y} t^{-\frac{5}{2}} \frac{dt}{t} \ll \int_0^{t_0} e^{-\frac{(\log 2)^2}{C_2 t}} \log Y_j t^{-\frac{5}{2}} \frac{dt}{t} \ll \log Y_j.$$

On the other hand, when $r \in [0, 2]$ we have $\ell(r) > cr$ for some $c > 0$ and thus:

$$(6.7) \quad \begin{aligned} \int_0^{t_0} \int_0^{Y_j} \int_{\alpha_n^j/y}^2 r^2 e^{-\frac{\ell(r)^2}{C_2 t}} dr \frac{dy}{y} t^{-\frac{5}{2}} \frac{dt}{t} &\ll \int_0^{Y_j} \int_0^{t_0} e^{-\frac{(\alpha_n^j/y)^2}{C_2 t}} t^{-\frac{5}{2}} \frac{dt}{t} \frac{dy}{y} \\ &\ll \int_0^{+\infty} e^{-\frac{1}{C_2 t} t^{-\frac{5}{2}}} \frac{dt}{t} \left(\frac{y}{\alpha_n^j} \right)^5 \frac{dy}{y} \\ &\ll \left(\frac{Y_j}{\alpha_n^j} \right)^5. \end{aligned}$$

We now define

$$(6.8) \quad Y_j = \max(\alpha_n^j \log \alpha_n^j, (\text{vol } M_n / h_n)^{\frac{1}{10}}),$$

we see that $\min_j (Y_j / \alpha_n^j) \xrightarrow{n \rightarrow \infty} 0$. For n large enough we have $(Y_j / \alpha_n^j) \geq \log Y_j$ for all $j = 1, \dots, h_n$, so that for all j (6.5), (6.6) and (6.7) above yield

$$\int_0^{t_0} \int_{D_n^Y} \sum_{\substack{\gamma \in \Gamma_n / (\Gamma_n)_{P_j} \\ \eta \in \Lambda_{n,j} - \{0\}}} t^{-\frac{3}{2}} e^{-\frac{d(x, \gamma \eta \gamma^{-1} x)^2}{C_2 t}} \frac{dt}{t} \ll \max((\log \alpha_n^j)^5, (\text{vol } M_n / h_n)^{\frac{1}{2}}).$$

Letting h_n^b be the smallest index j such that $\alpha_n^j \log \alpha_n^j \geq (\text{vol } M_n / h_n)^{\frac{1}{10}}$ we get:

$$\begin{aligned} \int_0^{t_0} \int_{M_n^{Y^n}} U_t(x) dx t^{-\frac{3}{2}} \frac{dt}{t} &\ll \sum_{j=1}^{h_n^b-1} \left(\frac{\text{vol } M_n}{h_n} \right)^{\frac{1}{2}} + \sum_{j=h_n^b}^{h_n} (\log \alpha_n^j)^5 \\ &\ll (h_n)^{\frac{1}{2}} (\text{vol } M_n)^{\frac{1}{2}} + \sum_{j=1}^{h_n} (\log \alpha_n^j)^5. \end{aligned}$$

The first summand is $o(\text{vol } M_n)$ by Lemma 4.2 and the second one by Lemma 4.6, which finishes the proof. \square

6.2.2. Comparisons.

Proposition 6.4. *Let $t_0 > 0$, $p = 1, 2, 3$ and Y^n the sequence from Proposition 6.3. We have*

$$(6.9) \quad \frac{1}{\text{vol } M_n} \frac{d}{ds} \left(\frac{1}{\Gamma(s)} \int_0^{t_0} (\text{Tr}_Y(e^{-t\Delta^p[M_n]}) - \text{Tr}(e^{-t\Delta_{\text{abs}}^p[M_n^{Y^n}]}) t^s \frac{dt}{t}) \right)_{s=0} \xrightarrow{n \rightarrow \infty} 0.$$

Proof. From (6.4) it follows that

$$\begin{aligned}
\mathrm{Tr}_{Y^n}(e^{-t\Delta^p[M_n]}) - \mathrm{Tr}(e^{-t\Delta_{\mathrm{abs}}^p[M_n^{Y^n}]}) &= \int_{D_n^{Y^n}} \mathrm{tr}(e^{-t\Delta[\mathbb{H}^3]}(x, x) - e^{-t\Delta_{\mathrm{abs}}^p[\widetilde{M_n^{Y^n}}]}(x, x))dx \\
&+ \int_{D_n^{Y^n}} \sum_{\gamma \in \Gamma_n - \{1\}} \mathrm{tr}(e^{-t\Delta^p[\mathbb{H}^3]}(x, \gamma x))dx \\
&+ \int_{D_n^{Y^n}} \sum_{\gamma \in \Gamma_n - \{1\}} \mathrm{tr}(e^{-t\Delta_{\mathrm{abs}}^p[\widetilde{M_n^{Y^n}}]}(x, \gamma x))dx \\
&=: E_1 + E_2 + E_3.
\end{aligned}$$

The manifolds $\widetilde{M_n^{Y^n}}$ are equal to $\widetilde{M^{Y^n}}$, so that the first summand $\frac{d}{ds}(\Gamma(s)^{-1} \int_0^{t_0} E_1 t^{s-1} dt)_{s=0}$ is equal to $\mathrm{vol} M_n (\log T^{(2)}(M; V) - \log T_{\mathrm{abs}}^{(2)}(\widetilde{M^{Y^n}}; V))$. It is proven in [12, Section 2] that

$$(6.10) \quad \log T^{(2)}(M; V) - \log T_{\mathrm{abs}}^{(2)}(\widetilde{M^Y}; V) \xrightarrow[\text{all } Y_j \rightarrow \infty]{} 0 \quad ^8$$

so that for any sequence Y^n with $\min Y_j^n \rightarrow \infty$ (in particular for those constructed in Proposition 6.3) we get

$$\frac{d}{ds} \left(\frac{1}{\Gamma(s)} \int_0^{t_0} E_1 t^{s-1} dt \right)_{s=0} \xrightarrow{n \rightarrow \infty} 0.$$

On the other hand Theorem 5.2 bounds yield

$$\mathrm{tr}(e^{-t\Delta_{\mathrm{abs}}^p[\mathbb{H}^3]}(x, \gamma x)), \mathrm{tr}(e^{-t\Delta_{\mathrm{abs}}^p[\widetilde{M^Y}]}(x, \gamma x)) \ll t^{-3/2} e^{-d(x, \gamma x)^2/C_2 t}$$

where the constant does not depend on Y , so that we have in the notation of Proposition 6.3 the inequality $E_2, E_3 \ll S(n, t, Y^n)$ and we get

$$\frac{d}{ds} \left(\frac{1}{\Gamma(s)} \int_0^{t_0} E_i t^{s-1} dt \right)_{s=0} \xrightarrow{n \rightarrow \infty} 0.$$

for $i = 2, 3$, which finishes the proof of the proposition. \square

Proposition 6.5. *Let Y^n be the sequence from Proposition 6.4, then we have*

$$\frac{1}{\mathrm{vol} M_n} \frac{d}{ds} \left(\frac{1}{\Gamma(s)} \int_0^{t_0} (\mathrm{Tr}_R e^{-t\Delta^p[M_n]} - \mathrm{Tr}_{Y^n} e^{-t\Delta^p[M_n]}) t^s \frac{dt}{t} \right)_{s=0} \xrightarrow{n \rightarrow \infty} 0.$$

Proof. From the explicit expression of the $o(1)$ in (3.18) we get for any $Y \in [1, +\infty)^{h_n}$:

$$\begin{aligned}
\mathrm{Tr}_Y \phi(\Delta^p[M_n]) - \mathrm{Tr}_R \phi(\Delta^p[M_n]) &= \sum_{j=1}^{h_n} 2\pi \log Y_j \int_0^{+\infty} r h_\phi^p(\ell(r)) dr \\
(6.11) \quad &+ \sum_{j=1}^{h_n} \mathrm{vol} \Lambda_{n,j} \int_0^{+\infty} r h_\phi^p(\ell(r)) \int_{\max(\alpha_n^j, rY_j)}^{+\infty} E_{\Lambda_{n,j}}(\rho) \frac{d\rho}{\rho^3} dr \\
&+ \sum_{j=1}^{h_n} \mathrm{vol} \Lambda_{n,j} \int_0^{+\infty} \frac{E_{\Lambda_{n,j}}(rY_j)}{(rY_j)^2} h_\phi^p(\ell(r)) dr.
\end{aligned}$$

⁸Proved there only for trivial coefficients, but the proof of their Theorem 2.26 works in general and the rest of the proof does not cause any problem.

We put

$$\begin{aligned}
T_1 &= \sum_{j=1}^{h_n} 2\pi \log Y_j \int_0^{+\infty} r h_t^p(\ell(r)) dr \\
T_2 &= \sum_{j=1}^{h_n} \text{vol } \Lambda_{n,j} \int_0^{+\infty} \frac{E_{\Lambda_{n,j}}(rY_j)}{(rY_j)^2} h_t^p(\ell(r)) dr. \\
T_3 &= \sum_{j=1}^{h_n} \text{vol } \Lambda_{n,j} \int_0^{+\infty} r h_t^p(\ell(r)) \int_{\max(\alpha_n^j, rY_j)}^{+\infty} \frac{E_{\Lambda_{n,j}}(\rho)}{\rho^3} d\rho dr
\end{aligned}$$

Recall that Θ was defined in (5.17), so that

$$\frac{d}{ds} \left(\frac{1}{\Gamma(s)} \int_0^{t_0} T_1 t^s \frac{dt}{t} \right)_{s=0} = \sum_{j=1}^{h_n} 2\pi \log Y_j \frac{d\Theta}{ds}(0)$$

and by the definition (6.8) of the sequence Y^n the right-hand side is an $O(\log(\text{vol } M_n) + \sum_{j=1}^{h_n} \log \alpha_n^j)$ which is itself an $o(\text{vol } M_n)$ by Lemma 4.6.

Recall the asymptotic expansion (5.6) :

$$h_t^p(\ell) = \sum_{k=0}^3 b_k^p(\ell) e^{-\frac{\ell^2}{C_2 t} t^{-\frac{k}{2}}} + O(t^{\frac{1}{2}}).$$

We obtain

$$\begin{aligned}
& \frac{d}{ds} \left(\frac{1}{\Gamma(s)} \int_0^{t_0} \sum_{j=1}^{h_n} \text{vol } \Lambda_{n,j} \int_0^{+\infty} \frac{E_{\Lambda_{n,j}}(rY_j)}{(rY_j)^2} e^{-\frac{\ell(r)^2}{C_2 t}} dr t^{s+\frac{1}{2}} \frac{dt}{t} \right)_{s=0} \\
&= \int_0^{t_0} \sum_{j=1}^{h_n} \text{vol } \Lambda_{n,j} \int_0^{+\infty} \frac{E_{\Lambda_{n,j}}(rY_j)}{(rY_j)^2} e^{-\frac{\ell(r)^2}{C_2 t}} dr t^{\frac{1}{2}} \frac{dt}{t} \\
&\ll \sum_{j=1}^{h_n} (\alpha_n^j Y_j)^{-1} \leq h_n
\end{aligned}$$

and it follows that

$$\begin{aligned}
& \frac{d}{ds} \left(\frac{1}{\Gamma(s)} \int_0^{t_0} T_2 t^s \frac{dt}{t} \right)_{s=0} \\
&= \sum_{j=1}^{h_n} \sum_{k=0}^3 \frac{d}{ds} \left(\frac{1}{\Gamma(s)} \text{vol } \Lambda_{n,j} \int_0^{+\infty} \frac{E_{\Lambda_{n,j}}(rY_j)}{(rY_j)^2} b_k^p(\ell(r)) e^{-\frac{\ell(r)^2}{C_2 t}} dr t^{s-\frac{k}{2}} \frac{dt}{t} \right)_{s=0} + O(h_n).
\end{aligned}$$

We study the integration in r on both sides of $r = \alpha_n^j / Y_j$. First we have

$$\begin{aligned}
\text{vol } \Lambda_{n,j} \int_{\alpha_n^j / Y_j}^{+\infty} \frac{E_{\Lambda_{n,j}}(rY_j)}{(rY_j)^2} b_k^p(\ell(r)) e^{-\frac{\ell(r)^2}{C_2 t}} dr &\ll (Y_j \alpha_n^j)^{-1} \int_{\alpha_n^j / Y_j}^{+\infty} r^{-1} b_k^p(\ell(r)) e^{-\frac{\ell(r)^2}{C_2 t}} dr \\
&\ll (Y_j \alpha_n^j)^{-1} e^{-\frac{(\alpha_n^j / Y_j)^2}{C_2 t}}
\end{aligned}$$

with a constant independant of n, j . Second:

$$\begin{aligned} \frac{\text{vol } \Lambda_{n,j}}{\pi} \int_0^{\alpha_n^j/Y_j} \frac{E_{\Lambda_{n,j}}(rY_j)}{(rY_j)^2} b_k^p(\ell(r)) e^{-\frac{\ell(r)^2}{C_2 t}} dr &= \int_0^{\alpha_n^j/Y_j} b_k^p(\ell(r)) e^{-\frac{\ell(r)^2}{C_2 t}} dr \\ &= \int_0^1 b_k^p(\ell(r)) e^{-\frac{\ell(r)^2}{C_2 t}} dr - \int_{\alpha_n^j/Y_j}^1 b_k^p(\ell(r)) e^{-\frac{\ell(r)^2}{C_2 t}} dr. \end{aligned}$$

We have $\int_{\alpha_n^j/Y_j}^{+\infty} b_k^p(\ell(r)) e^{-\frac{\ell(r)^2}{C_2 t}} dr \ll e^{-\frac{(\alpha_n^j/Y_j)^2}{C_2 t}}$, and moreover (5.8) yields the expansion

$$\int_0^1 b_k^p(\ell(r)) e^{-\frac{\ell(r)^2}{C_2 t}} dr = \sum_{l=2}^m c_l t^{l/2}$$

with coefficients c_l not depending on n, j . It follows that

$$\text{vol } \Lambda_{n,j} \int_0^{\alpha_n^j/Y_j} \frac{E_{\Lambda_{n,j}}(rY_j)}{(rY_j)^2} b_k^p(\ell(r)) e^{-\frac{\ell(r)^2}{C_2 t}} dr = \pi \sum_{l=2}^m c_l t^{l/2} + O(e^{-\frac{(\alpha_n^j/Y_j)^2}{C_2 t}})$$

Gathering all estimates we get:

$$\frac{d}{ds} \left(\frac{1}{\Gamma(s)} \int_0^{t_0} T_2 t^s \frac{dt}{t} \right)_{s=0} \ll h_n + \sum_{j=1}^{h_n} \sum_{k=0}^3 (1 + (\alpha_n^j Y_j)^{-1}) \int_0^{t_0} e^{-\frac{(\alpha_n^j/Y_j)^2}{C_2 t}} t^{-\frac{k}{2}} \frac{dt}{t}.$$

The second term on the right is dealt with as in the proof of Proposition 6.3: for $k = 1, 2, 3$ we have

$$\int_0^{t_0} e^{-\frac{(\alpha_n^j/Y_j)^2}{C_2 t}} t^{-\frac{k}{2}} \frac{dt}{t} \leq \left(\frac{Y_j}{\alpha_n^j} \right)^k \int_0^{+\infty} e^{-1/C_2 t} t^{-\frac{k}{2}} \frac{dt}{t}$$

and for $k = 0$

$$\int_0^{t_0} e^{-\frac{(\alpha_n^j/Y_j)^2}{C_2 t}} \frac{dt}{t} = \int_0^{\frac{Y_j}{\alpha_n^j} t_0} e^{-1/C_2 t} \frac{dt}{t} \ll \log(Y_j/\alpha_n^j)$$

so that we finally obtain

$$\frac{d}{ds} \left(\frac{1}{\Gamma(s)} \int_0^{t_0} T_2 t^s \frac{dt}{t} \right)_{s=0} \ll \sum_{j=1}^{h_n} \left(\frac{Y_j}{\alpha_n^j} \right)^3$$

which goes to 0 as $n \rightarrow \infty$ for $Y = Y^n$.

The summand T_3 is dealt with in a similar manner. For any $n \geq 1$ and $j = 1, \dots, h_n$ we put

$$\kappa'_j = \text{vol } \Lambda_{n,j} \int_{\max(\alpha_n^j, rY_j)}^{+\infty} E_{\Lambda_{n,j}}(\rho) \frac{d\rho}{\rho^3},$$

We have seen in the proof of Theorem 4.4 that κ'_j is bounded independantly of n, j and we get

$$\begin{aligned} \text{vol } \Lambda_{n,j} \int_0^{+\infty} r h_t^p(\ell(r)) \int_{\max(\alpha_n^j, rY_j)}^{+\infty} E_{\Lambda_{n,j}}(\rho) \frac{d\rho}{\rho^3} dr \\ = \kappa'_j \int_0^{\alpha_n^j/Y_j} r h_t^p(\ell(r)) dr + \int_{\alpha_n^j/Y_j}^{+\infty} r h_t^p(\ell(r)) \int_{\max(\alpha_n^j, rY_j)}^{+\infty} E_{\Lambda_{n,j}}(\rho) \frac{d\rho}{\rho^3} dr \\ = \kappa'_j \int_0^1 r h_t^p(\ell(r)) dr + O(t^{-\frac{3}{2}} e^{-\frac{(\alpha_n^j/Y_j)^2}{C_2 t}}). \end{aligned}$$

Thus, we have

$$\frac{d}{ds} \left(\frac{1}{\Gamma(s)} \int_0^{t_0} T_3 t^s \frac{dt}{t} \right)_{s=0} \ll h_n + \sum_{j=1}^{h_n} \int_0^{t_0} e^{-\frac{(\alpha_n^j/Y_j)^2}{Ct}} t^{-\frac{3}{2}} \frac{dt}{t}$$

and we have already proved that the right-hand side is an $o(\text{vol } M_n)$, which concludes the proof. \square

The following result is an immediate consequence of Propositions 6.4 and 6.5.

Corollary 6.6. *For the sequence Y^n from Proposition 6.3 we have*

$$\lim_{n \rightarrow \infty} \frac{\log T_R(M_n; V) - \log T_{\text{abs}}(M_n^{Y^n}; V)}{\text{vol } M_n} = 0.$$

6.3. Large-time asymptotics.

Proposition 6.7. *For the sequence Y^n from proposition 6.4 we have that*

$$\sup_n \left(\int_{t_0}^{+\infty} \text{Tr}(e^{-t\Delta_{\text{abs}}^p[M_n^{Y^n}]}) \frac{dt}{t} \right)$$

is finite and goes to 0 as $t_0 \rightarrow \infty$.

6.3.1. Spectral gap for the truncated manifolds. Here we state in a more precise form some of the results of [7, Chapter 6]. There are constants C, c depending on Γ such that for any n and any eigenform $f \in L^2\Omega^p(M; V)$ with eigenvalue λ we have for all $Y \in [1, +\infty)^{h_n}$ such that $Y_j/\alpha_n^j \geq C\sqrt{\lambda}$ the following bounds, making [7, 6.1.3] more precise:

$$(6.12) \quad |f(x) - f_P(x)| \leq |f|_{L^2\Omega^p(M_n^{C\lambda}; V)} e^{-c \frac{y_j(x)}{\alpha_n^j}} \text{ for all } x \in M_n - M_n^Y.$$

Let $f_1(s) = \det(1 - Y^{-4s}\Phi(s))$, $f_0(s) = \det(1 - Y^{-2s}\Psi(s))$ and $\mathcal{N}_f(u) = |\{s \in i[0, u], f(s) = 0\}|$. Let $\mathcal{N}_Y^1(u)$ be the number of eigenvalues of $\Delta_{\text{abs}}^p[M_n^Y]$ and $\mathcal{N}_{\text{cusp}}^1(u)$ the number of eigenvalues of $\Delta_{\text{cusp}}^p[M_n]$ in the interval $[0, u^2]$. Using the bounds (6.12) above in the proof of [7, Theorem 6.12] we get the following result.

Theorem 6.8. *Let $\mathcal{N}_M^1 = \mathcal{N}_{\text{cusp}}^1 + \mathcal{N}_{f_1} + \mathcal{N}_{f_0}$. There are $c, C > 0$ such that for all n and Y such that $\forall j, Y_j \geq C\alpha_n^j$ we have for all λ such that $\lambda^2 \leq (Y_j/\alpha_n^j)^2$:*

$$(6.13) \quad \mathcal{N}_{M_n}^1(\sqrt{\lambda} - e^{-c \frac{Y}{\alpha_n^j}}) \leq \mathcal{N}_Y^1(\sqrt{\lambda}) \leq \mathcal{N}_{M_n}^1(\sqrt{\lambda} + e^{-c \frac{Y}{\alpha_n^j}})$$

The same theorem holds for sections of V and 2- and 3- forms, replacing \mathcal{N}^1 by the appropriate function (see [7]; in the case of strongly acyclic coefficients the intertwining operator Ψ has no pole at $s = 1$ so that there are no “monster eigenvalues”, as Calegari et Venkatesh call them). We get the following corollary (recall that $\lambda_0 > 0$ is a lower bound for the spectra with coefficients in V of hyperbolic manifolds).

Corollary 6.9. *There exists a constant C (depending on Γ) such that for any n and Y such that $Y_j \geq C\alpha_n^j$ the smallest eigenvalue of $\Delta_{\text{abs}}^p[M_n^Y]$ is larger than $\lambda_0/2$.*

6.3.2. Conclusion.

Proof of Proposition 6.7. We proceed exactly as in 5.4.2 above, but have to check that both properties of the heat kernel used there are still verified by the $M_n^{Y^n}$. Namely we need to show that

(i) There is a λ_0^{abs} such that for all n and $f \in L^2\Omega_{\text{abs}}^p(M_n^{Y^n}; V)$ we have

$$\langle \Delta_{\text{abs}}^p f, f \rangle_{L^2\Omega^p(M_n^{Y^n}; V)} \geq \lambda_0^{\text{abs}}.$$

(ii) The sequence $\text{Tr}(e^{-\Delta_{\text{abs}}^p[M_n^{Y^n}]})$ is bounded.

The point (i) is a direct consequence of Corollary 6.9 above since we have that $\min_j(Y_j^n/\alpha_n^j) \rightarrow \infty$ as $n \rightarrow \infty$.

We deduce (ii) from the following more precise result: for all $t > 0$ we have in fact the limit

$$(6.14) \quad \lim_{n \rightarrow \infty} \frac{\text{Tr} e^{-t\Delta_{\text{abs}}^p[M_n^{Y^n}]}}{\text{vol } M_n} = \text{Tr}_\Gamma(e^{-t\Delta^p[\mathbb{H}^3]})$$

Indeed, we see that

$$\frac{|\text{Tr}(e^{-t\Delta_{\text{abs}}^p[M_n^{Y^n}]} - \text{Tr}_\Gamma(e^{-t\Delta^p[\mathbb{H}^3]}))|}{\text{vol } M_n} \ll |\text{Tr}_\Gamma(e^{-t\Delta_{\text{abs}}^p[\widetilde{M^{Y^n}}]} - \text{Tr}_\Gamma(e^{-t\Delta^p[\mathbb{H}^3]}))| + \frac{S(n, t, Y^n)}{\text{vol } M_n}.$$

The proof of Proposition 6.3 yields that $S(n, t, Y_n) = o(\text{vol } M_n)$ and (6.10) that the first summand also goes to 0 as $n \rightarrow \infty$. \square

6.4. An asymptotic Cheeger-Müller equality. We give here the proof of Theorem 6.2. For a finite-volume hyperbolic manifold M let g_0 be the hyperbolic metric on M^Y and g_1 a Riemannian metric on M^Y which equals g_0 on $M^{Y/3}$ and is a product on a neighbourhood of the boundary, for example we can take

$$g_1(x, y) = dy^2 + \psi(\log(Y/y))Y^{-2} + (1 - \psi(\log(Y/y)))y^{-2}$$

where ψ is a smooth function which is zero on $[1, +\infty)$ and constant equal to 1 near zero. we put $g_u = ug_1 + (1-u)g_0$ which is a smooth family of Riemannian metrics on M^Y . The following result is well-known.

Proposition 6.10. *There exists smooth functions $c_p(u)$ depending only on ψ such that we have, for all n and $Y \in [1, +\infty)^{h_n}$:*

$$\frac{d}{du} (\log T_{\text{abs}}(M_n^Y, g_u) - \log \tau_{\text{abs}}(M_n^Y, g_u)) = \text{vol}(\partial M_n^Y) \sum_{p=0}^3 p(-1)^p c_p(u).$$

Proof. This follows at once from [8, Theorem 3.27] since the germ of g_u on the boundary ∂M_n^Y does not depend on n or Y . \square

On the other hand, by [6, Theorem 0.1] we get that $T_{\text{abs}}(M_Y, g_1) = \tau_{\text{abs}}(M_Y, g_1)$ so that

$$\frac{\log T_{\text{abs}}(M^Y) - \log \tau_{\text{abs}}(M^Y)}{\text{vol } M} = \text{vol}(\partial M^Y) \int_0^1 \sum_{p=1}^3 (-1)^p p c_p(u) du$$

and since for the sequence Y_n from Proposition 6.3 we have

$$\text{vol}(\partial M_n^{Y^n}) \ll \sum_{j=1}^{h_n} Y_j^{-2} [\Lambda_j : \Lambda_{n,j}] \leq (\min_j Y_j^n)^{-2} \text{vol } M_n = o(\text{vol } M_n)$$

it follows that

$$\frac{\log T_{\text{abs}}(M_n^{Y^n}; V) - \log \tau_{\text{abs}}(M_n^{Y^n}; V_{\mathbb{Z}})}{\text{vol } M_n} \xrightarrow{n \rightarrow \infty} 0.$$

6.5. Growth of Betti numbers. The following result is a generalization (for $G = \mathrm{SL}_2(\mathbb{C})$) of Corollary 1.3 in [1] to the noncompact setting. Note that we could not deduce it immediately from the convergence of the regularized trace because of the spectral terms coming from the Eisenstein series.

Corollary 6.11. *Let M_n be a cusp-uniform sequence of finite covers of an hyperbolic three-manifold and suppose that M_n BS-converges to \mathbb{H}^3 . Then we have for $p = 1, 2$*

$$(6.15) \quad \frac{b_p(M_n)}{\mathrm{vol}(M_n)} \xrightarrow{|\mathcal{I}| \rightarrow \infty} 0$$

Proof. For all n and $t > 0$ we have $\dim \ker(\Delta_{\mathrm{abs}}^p[M_n^{Y^n}]) \leq \mathrm{Tr} e^{-t\Delta_{\mathrm{abs}}^p[M_n^{Y^n}]}$. On the other hand we have $b_p(M_n) = \dim \ker(\Delta_{\mathrm{abs}}^p[M_n^{Y^n}])$ and it follows that for any $t > 0$ we have:

$$\limsup_{n \rightarrow \infty} \frac{b_p(M_n)}{\mathrm{vol} M_n} \leq \lim_{n \rightarrow \infty} \frac{\mathrm{Tr} e^{-t\Delta_{\mathrm{abs}}^p[M_n^{Y^n}]}}{\mathrm{vol} M_n} = \mathrm{Tr}_{\Gamma} e^{-t\Delta^p[\mathbb{H}^3]}.$$

The right-hand side goes to 0 as $t \rightarrow \infty$ since $b_k^{(2)}(\mathbb{H}^3) = 0$ (cf. [14, Theorem 1.63]) and (6.15) follows. \square

This limit is well-known for exhaustive sequences as follows for example from Lück's theorem [11] (applied to the manifolds truncated at 1). The result for principal congruence subgroups is due to Savin [24] for a general simple group, and for $\mathrm{SL}_2(\mathbb{C})$ Sarnak and Xue [23] give sublinear upper bounds for principal congruence subgroups.

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